A universal framework for hydrodynamics

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ABSTRACT: In this thesis, we present a universal framework for hydrodynamics starting from the fundamental considerations of symmetries and the second law of thermodynamics, while allowing for additional gapless modes in the low-energy spectrum. Examples of such fluids include superfluids and fluids with surfaces. Typically, additional dynamical modes in hydrodynamics also need to be supplied with their own equations of motion by hand, like the Josephson equation for superfluids and the Young-Laplace equation for fluid surfaces. However, we argue that these equations can be derived within the hydrodynamic framework by a careful off-shell generalisation of the second law. This potentially provides a universal framework for a large class of hydrodynamic theories, based on their underlying symmetries and gapless modes. Motivated by this newly found universality, we present an all-order analysis of the second law of thermodynamics and propose a classification scheme for the allowed hydrodynamic transport, including arbitrary gapless modes, independent spin current, and background torsion.

In the second half of this thesis, we look at the construction of null fluids which are a new viewpoint of Galilean fluids. These are essentially fluids coupled to spacetime backgrounds carrying a covariantly constant null isometry, but with additional constraints imposed on the background gauge field and affine connection to reproduce the correct Galilean degrees of freedom. We discuss the Galilean version of quantum anomalies and their effect on hydrodynamics. Finally, we follow our relativistic discussion to allow for arbitrary gapless modes in Galilean hydrodynamics and present a classification scheme for the second law abiding hydrodynamic transport at all orders in the derivative expansion.

We apply these abstract ideas to review the theory of ordinary relativistic/Galilean hydrodynamics and provide novel constructions for relativistic/Galilean (non-Abelian) superfluid dynamics and surface transport. We also comment on the possible application to the theory of magnetohydrodynamics.

Dedicated to my parents

Smt. Neelam Jain & Sh. Neeraj Jain

for shaping me into who I am

and to my brother

Abhinav

for helping along the way

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. The results presented here are derived from the following independent and collaborative works:

- N. Banerjee, S. Dutta and A. Jain, *Equilibrium partition function for nonrelativistic fluids*, *Phys. Rev.* **D92** (2015) 081701, [1505.05677].
- N. Banerjee, S. Dutta and A. Jain, *Null Fluids - A New Viewpoint of Galilean Fluids*, *Phys. Rev.* **D93** (2016) 105020, [1509.04718].
- A. Jain, *Galilean Anomalies and Their Effect on Hydrodynamics*, *Phys. Rev.* **D93** (2016) 065007, [1509.05777].
- A. Jain, *Theory of non-Abelian superfluid dynamics*, *Phys. Rev.* **D95** (2017) 121701, [1610.05797].
- N. Banerjee, S. Dutta and A. Jain, *First Order Galilean Superfluid Dynamics*, *Phys. Rev.* **D96** (2017) 065004, [1612.01550].
- J. Armas, J. Bhattacharya, A. Jain and N. Kundu, *On the surface of superfluids*, *JHEP* **06** (2017) 090, [1612.08088].

While this thesis was being prepared, the following collaborative works were also published by the author, results from which have not been included in this thesis:

- N. Banerjee, S. Atul Bhatkar and A. Jain, *Second order Galilean fluids & Stokes' law*, *Phys. Rev.* **D97** (2018) 096018, [1711.09076].
- J. Armas, J. Gath, A. Jain and A. V. Pedersen, *Dissipative hydrodynamics with higher-form symmetry*, *JHEP* **05** (2018) 192, [1803.00991].
- P. Burda, R. Gregory and A. Jain, *Holographic Reconstruction of Bubbles*, 1804.05202.

No part of this thesis has been submitted for a degree or qualification in this or any other institution.

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Contents

Notation and conventions

Spacetime and indices

We use mostly positive metric convention throughout this work. Number of physical spacetime dimensions are denoted by d. The Greek letters $(\mu, \nu, \ldots)$ are used to denote the d-dimensional spacetime indices, while the Greek letters $(\alpha, \beta, \ldots)$ are used for the indices on the d-dimensional frame bundle. Similarly, the Roman letters $(i, j, \ldots)$ denote the $(d-1)$-dimensional spatial indices, with the corresponding frame bundle indices denoted by the Roman letters $(a, b, \ldots)$. The time coordinate, on the other hand, is denoted by t. We also introduce $(d+1)$-dimensional null backgrounds as a tool for Galilean hydrodynamics. Indices on it are denoted by the Roman letters $(\mathrm{M}, \mathrm{N}, \ldots)$, while those on its frame bundle by $(\mathrm{A}, \mathrm{B}, \ldots)$. In the context of anomaly inflow mechanism, the "bulk" indices will be denoted by a hat. Einstein summation convention is implied everywhere.

We use the round and square brackets to denote totally symmetric and totally anti-symmetric combinations of a tensor respectively. For a 2-tensor, $M^{\mu\nu} = M^{(\mu\nu)} + M^{[\mu\nu]}$. We often also use the angular brackets to denote traceless symmetric combinations.

Differential forms

We denote differential forms by bold-faced characters. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are m and n-rank differential forms respectively, X^μ is a vector field, $f(x)$ is a function, and $g = \det g_{\mu\nu}$ is the metric determinant, then

$$
\begin{aligned}
\boldsymbol{A} &= \frac{1}{m!} A_{\mu_1 \ldots \mu_m} \mathrm{d}x^{\mu_1} \wedge \ldots, \\
\boldsymbol{\epsilon} &= \sqrt{|g|}\ \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^d && \text{[Volume form]} \\
\boldsymbol{A} \wedge \boldsymbol{B} &= \frac{1}{(m+n)!} \left(\frac{(m+n)!}{m!n!} A_{[\mu_1 \ldots \mu_m} B_{\nu_{m+1} \ldots \nu_{m+n}]} \right) \mathrm{d}x^{\mu_1} \wedge \ldots, && \text{[Exterior product]} \\
\star\boldsymbol{A} &= \frac{1}{(d-m)!} \left(\frac{1}{m!} A^{\mu_1 \ldots \mu_m} \epsilon_{\mu_1 \ldots \mu_m \nu_1 \ldots \nu_{d-m}} \right) \mathrm{d}x^{\nu_1} \wedge \ldots, && \text{[Hodge dual]} \\
\star\star\boldsymbol{A} &= \mathrm{sgn}(g) (-)^{m(d-m)}, \\
\iota_X \boldsymbol{A} &= \frac{1}{(m-1)!} \left(X^\mu A_{[\mu\nu_1 \ldots \nu_{m-1}]} \right) \mathrm{d}x^{\nu_1} \wedge \ldots, && \text{[Interior product]} \\
\mathrm{d}\boldsymbol{A} &= \frac{1}{(m+1)!} \left((m+1) \partial_{[\mu_1} A_{\mu_2 \ldots \mu_{m+1}]} \right) \mathrm{d}x^{\mu_1} \wedge \ldots, && \text{[Exterior derivative]} \\
\pounds_X \boldsymbol{A} &= \iota_X \mathrm{d}\boldsymbol{A} + \mathrm{d}\left(\iota_X \boldsymbol{A} \right), && \text{[Lie derivative]} \\
\int f(x) \boldsymbol{\epsilon} &= \int \mathrm{d}^d x \sqrt{|g|}\, f(x). && \text{[Integration]}
\end{aligned}
$$

1 | Introduction

Hydrodynamics, or fluid dynamics, is a scientific discipline which concerns itself with the study of fluids. The word "hydrodynamics" derives from the Greek prefix "hydro-" meaning "water" and literally translates to "the study of motion of water". The word "fluid", on the other hand, comes from the Latin word "fluere" meaning "to flow", and is used as an umbrella term for liquids and gases together. Today, at least within the high energy physics community, both hydrodynamics and fluid dynamics are used interchangeably to describe the physics of continuum systems ranging from the infinitesimal quark-gluon plasma to the infinite universe itself.

Over seventy per cent of Earth's surface is covered with water, which undoubtedly contributes the most vital ingredient to our biology, chemistry, as well as our sociology. Major civilisations in the history of humankind have flourished around water bodies, from ancient Mesopotamia all the way to present-day London. Understanding the physics of water, or more generally of fluids, has therefore played a key role in shaping the human civilisation. The earliest records of scientific enquiry into the statics and dynamics of fluids date back to Archimedes in his treatise *On Floating Bodies*, published around 250 BC. However, it is generally accepted that a pragmatic, if not scientific, understanding of fluids existed even in ancient human civilisations. Over the two millennia following Archimedes' principle of buoyancy, the field of hydrodynamics attracted the attention of many inquisitive minds, slowly taking shape into a scientific discipline in its own right. Rapid progress was made in the seventeenth and eighteenth centuries alongside the development of calculus and the laws of thermodynamics, to which we owe much of our current understanding of hydrodynamics. Major contributors include Torricelli, Newton, Pascal, Bernoulli, Euler, d'Alembert, Lagrange, Laplace, Poisson, Poiseuille, Hagen, Navier, Stokes, Prandtl, Reynolds, and Taylor to name a few. In the early twentieth century, principles of hydrodynamics were reconciled with Einstein's theory of relativity paving way to our modern understanding of accelerator physics and cosmology. Exotic phenomena in hydrodynamics like dissipation, vorticity, turbulence, diffusion, superfluidity, and multifluid models, and their relation to thermodynamics and statistical mechanics were also developed in the twentieth century owing to the efforts of Ertel, Lichnerowicz, Kamerlingh-Onnes, Landau, Onsager, Prigogine, Khalatnikov, Eckart, Carter, Israel, and Stewart among many others. The state of the art in the field is the standard book by Landau and Lifshitz on *Fluid Mechanics* [10] published in 1959. To this day, hydrodynamics continues to be one of the most active topics of research in many areas of science including engineering, condensed matter physics, medicine, biophysics, and even high energy physics.

Conventionally, hydrodynamics has been an empirical field of study under the jurisdiction of material sciences and had little to do with the realm of high energy physics, which concerns itself with the fundamental machinery of our universe. However, this began to change towards the end of the twentieth century, when a series of advancements in string theory intricately wove hydrodynamics with the physics of black holes. String theory is a candidate for the quantum theory of gravity, which aims to describe all the four forces of nature within a grand unified framework. Three of these forces: electromagnetic, strong, and weak, have already been unified into a quantum field theory called the standard model of particle physics, with great experimental success at particle accelerator experiments such as the Large Hadron Collider. On the other hand, our current understanding of the fourth fundamental force, gravity, comes from a classical field theory known as the general theory of relativity. It has also been experimentally tested to great lengths, with its latest confirmation coming from the recent gravitational wave detection by LIGO and VIRGO collaborations in 2015 [11]. Despite these success stories and decades of research efforts, quantising gravity and reconciling it with the standard model of particle physics has proven to be very hard. This is where string theory comes in, with its revolutionary idea that the fundamental building blocks of our universe are not various species of point-like particles, as proposed by the standard model, but just a single species of tiny strings. In string theory, point-like particles are understood as emerging from different oscillatory modes of the same fundamental string. A quantum theory of strings gives us a natural framework where gravity and quantum field theories can, in principle, emerge from the same fundamental principles. Whether or not string theory is the answer to life, the universe and everything, it has already taught us some valuable lessons about nature. Perhaps the most important of these lessons is the concept of *holography*. It is the observation that the physics of quantum gravity in spacetimes with a certain asymptotic structure can be equivalent to that of a one-lower dimensional quantum field theory living at its boundary [12, 13]; see [14] for a review. The most popular example of holography comes from the *AdS/CFT correspondence* [15–17], which relates quantum gravity on asymptotically Anti-de Sitter (AdS) spacetimes in the bulk to conformal field theories at the boundary. See [18] for an introductory review.

Holographic dualities have proven to be invaluable for both quantum gravity as well as field theory research. On the one hand, they relate quantum field theories in a weak coupling regime to highly quantum phenomena on the gravity side and can be used to get valuable insights into the quantum nature of gravity. For example, it is widely felt that spacetime itself could be seen as emerging from the entanglement structure of the boundary field theory [9, 19]. More generally, there seems to be an intricate relationship between quantum gravity in the bulk and quantum information theory at the boundary [20–23], which can be used to probe hard questions in gravity like the black hole information paradox [24]. On the other hand, holography also relates classical general relativity to strongly coupled quantum field theories. The strong coupling regime can be very hard to access directly in quantum field theories due to the breakdown of the conventional perturbative approach using Feynman diagrams. Interestingly, the theory of strong interactions between quarks and gluons, called quantum chromodynamics (QCD), is weakly coupled at high energies but is strongly coupled at our day-to-day energy scales. Due to this feature of QCD, many strong interaction phenomena remain very hard to describe theoretically. Although novel techniques are being developed to directly tackle strongly coupled phenomena in quantum field theories, like conformal bootstrap [25] and lattice field theories [26], holography is one of our best analytical techniques to gain insights into these problems. This has led to a whole new field of research called *holographic condensed matter physics*, aimed at utilising holographic techniques to study qualitative features of real-world condensed matter systems. See for example [27] for an extensive review.

A particular niche of holography, called the *fluid/gravity correspondence*, dualises classical perturbations around black hole horizons in the bulk to a strongly-coupled fluid living at the boundary. The fact that the linearised Einstein equations governing black hole fluctuations can be remapped to the linearised dynamical equations of hydrodynamics, had been known since the late 1980s under the name of *membrane paradigm* [28–30]. But the full non-linear incarnation of this correspondence was only made explicit for the first time in 2007 [31]. Due to this correspondence, every solution to hydrodynamic equations of motion at the boundary gives rise to a dynamical black hole solution in the bulk. This has been a powerful tool in the literature, for example, to test the stability of black hole configurations using our intuitive notion of stability in hydrodynamics. In this language, the well known Gregory-Laflamme instability in black strings [32] can be holographically understood as the Rayleigh-Plateau instability in hydrodynamics [33, 34], which explains why a thin stream of fluid tends to break into smaller packets. Reversing the duality, the fluid/gravity correspondence has also led to some novel insights into hydrodynamics itself, and its strongly-coupled applications like the quark-gluon plasma [35, 36]. For instance, it was experimentally discovered that the ratio of shear viscosity to entropy density for a quark-gluon plasma takes a universal value of $\hbar/4\pi k_{\mathrm{B}}$ [37], which can be theoretically explained by holographic models [38, 39]. In fact, actual heavy-ion collisions at particle accelerator experiments can themselves be modelled holographically using the fluid/gravity correspondence [40, 41]. On other fronts, one of the most interesting aspects of this correspondence has been to provide a test-bed for new models of hydrodynamic phenomena, by providing an analytically manageable tool for otherwise intractable computations. Some examples involve recent holographic studies into exotic phenomena in hydrodynamics like vortices, turbulence and chaos [42–45].

These holographic explorations have also fuelled new research into hydrodynamics in its own right, leading to new insights into its fundamental principles and applications. Perhaps the most important of these insights have been realising the signatures of quantum anomalies in hydrodynamics [46, 47], which in turn have found crucial applications in the physics of neutron stars [48]. We also have a microscopically motivated derivation of a large sector of hydrodynamics using hydrostatic partition functions [2, 49–51], along with a classification of entire hydrodynamic transport to all orders in derivative expansion [1, 52]. Other important advancements include new models for superfluid dynamics [1, 53], magnetohydrodynamics [54], boost non-invariant [55, 56] and translation non-invariant hydrodynamics [57, 58], formation of surfaces and lumps [6, 59], and hydrodynamics with higher-form symmetries [8, 60]. Novel applications of hydrodynamics to condensed matter systems like high-temperature superconductors [61] and graphene [62] have also been realised recently. However, there are still a lot of fundamental questions that remain unanswered. For instance, we still do not satisfactorily understand how inherently reversible microscopic field theories can lead to irreversible phenomena, such as dissipation and the second law of thermodynamics, in the macroscopic limit. Also, our understanding of non-equilibrium processes in finite-temperature field theories is not very robust, with hydrodynamics practically being the only laboratory where we have some degree of control. The hope is that if we can understand how hydrodynamics emerges as an effective field theory from the microscopic degrees of freedom, we will be able to, at least, frame these questions in the right language. Considerable progress has been made towards a Schwinger-Keldysh formalism for hydrodynamics and writing down a Wilsonian effective action [63–71], which sheds light on some of these fundamental issues.

On an independent front, our understanding of Galilean hydrodynamics has also been revolutionised over the past decade. The holographic correspondence for Galilean (Schrödinger) hydrodynamics was first set up in 2008 [72], shortly after its relativistic counterpart. The authors employed the standard prescription of null reduction [73–75] (see [76] for a review), which reduces a $(d+1)$-dimensional relativistic fluid to a d-dimensional Galilean fluid, and prescribed a holographic map between d-dimensional Galilean hydrodynamics and $(d+2)$-dimensional black holes. Exploiting null reduction, a new framework for Galilean fluids has been proposed, called null fluids, which reorganises Galilean hydrodynamics into one-higher dimensional anisotropic relativistic hydrodynamics [2–5, 7]. Other frameworks of Galilean hydrodynamics, utilising Newton-Cartan geometries [77–80], have also been proposed recently [81, 82]. Besides Galilean, there have also been other incarnations of non-relativistic hydrodynamics in the literature with different underlying symmetry groups. Most notably, Lifshitz hydrodynamics has attracted a lot of attention; see e.g. [55, 83] for some recent work and [84] for a review of Lifshitz holography.

We should mention that the references provided here are only intended to be indicative of the ongoing activity in the field of hydrodynamics. Given the sheer magnitude of relevant papers being published every year, the references mentioned here are in no way exhaustive, or even representative, of all the interesting research being carried out in hydrodynamics and related areas. The readers interested in learning more are encouraged to consult the mentioned references and follow the bibliographies therein.

In this thesis, we review some of the recent advancements mentioned above to which the author has contributed during the course of his PhD [1–6]. We also discuss some new results and insights following from these works, which were found during the preparation of this thesis. We present a universal framework for hydrodynamics starting from the fundamental considerations such as symmetries and the second law of thermodynamics, while allowing for arbitrary gapless modes in the low-energy spectrum. The most natural examples of such systems are (non-Abelian) superfluids, which we discussed in [1], with gapless Goldstone modes arising due to a spontaneously broken internal symmetry. Such modes can also arise due to a spontaneous or explicit breaking of spacetime symmetries, which can cause the formation of surfaces, as we described in [6], and potentially a myriad of other interesting phenomena like momentum relaxation and boost non-invariance. In fact, the theory of magnetohydrodynamics can also be understood within the same framework, with the components of a dynamical U(1) gauge field serving as gapless modes. Typically, such additional dynamical modes in hydrodynamics also need to be supplied with their own equations of motion by hand, like the Josephson equation for superfluids, Young-Laplace equation for fluid surfaces, and Maxwell's equations for magnetohydrodynamics. However, we realised in [1] that these equations can be derived within the hydrodynamic framework by a careful off-shell generalisation of the second law of thermodynamics. This potentially provides a universal framework for a large class of hydrodynamic theories, based on their underlying symmetries and gapless modes. Motivated by this newly found universality, we extend the classification scheme of hydrodynamic transport, proposed by the authors of [52], to include arbitrary gapless modes, while also allowing for an independent spin-current and background torsion. This classification scheme first appeared for (non-Abelian) superfluids in [1]. We expect this construction to naturally extend to hydrodynamics with higher form symmetries as well, which we set up in [8], but we do not explore this direction in this thesis. In the second half of this thesis, we look at the construction of null fluids, based on our work in [2–5], which are a new viewpoint of Galilean fluids. These are essentially fluids coupled to spacetime backgrounds carrying a covariantly constant null isometry, but with additional constraints imposed on the background gauge field and affine connection to reproduce the correct Galilean degrees of freedom. We discuss the Galilean version of quantum anomalies and their effect on hydrodynamics, taken from our work in [5]. Finally, we follow our relativistic discussion to allow for arbitrary gapless modes in Galilean hydrodynamics and present a classification scheme for the second law abiding hydrodynamic transport at all orders in the derivative expansion. As far as we are aware, an all-order analysis of Galilean transport has not appeared in the literature before.

In the remainder of this introductory chapter, we present a quick recap of the fundamentals of hydrodynamics in section 1.1, followed by an introduction to hydrodynamics with gapless modes in section 1.2. Finally, in section 1.3 we provide a comprehensive overview of the main results of this thesis.

1.1 Fundamentals of hydrodynamics

Hydrodynamics is the low-energy effective description of a generic finite temperature quantum system near its thermodynamic equilibrium. The qualifier "near" essentially means that we are only ever allowed to leave the global thermodynamic equilibrium perturbatively so that locally we still have a notion of thermodynamic equilibrium at every spacetime point. In precise terms, hydrodynamics describes physical systems whose fluctuations are on the length scales much larger compared to the inherent length scales of the system, like mean-free path. Even in the absence of a (quasi-)particle description, temperature itself can set such an inherent length scale, $\beta = c\hbar/k_{\rm B}T$, which for a system at room temperature is roughly 10^{-6} meters. Within this narrow regime of applicability, hydrodynamics is quite universal; the fundamental equations governing it can be applied to water at the scale of a raft, quark-gluon plasma at the scale of particle accelerators, and even cosmology at the scale of the universe itself. In this section, we build hydrodynamics starting from the fundamental laws of thermodynamics. We briefly discuss both the relativistic and Galilean versions of hydrodynamics and work out the first corrections to the respective constitutive relations as we leave the thermodynamic equilibrium. This should set the groundwork for the material presented in this thesis.

1.1.1 Thermodynamics

Let us say that we are given a volume of gas in equilibrium and we want to describe it using our understanding of thermodynamics. The macroscopic variables we should choose to describe this system are based on the *statistical ensemble* we are working in. For example, if the gas is held in a closed container of fixed volume and is not allowed to exchange heat or conserved charges (like particles) with its surroundings, it is said to be in the microcanonical ensemble. The thermodynamic parameters in this ensemble are the volume V of the container and the total entropy S_{tot} and total charge Q_{tot} of the gas. Their canonical conjugates, on the other hand, are the thermodynamic observables: pressure P, temperature T, and chemical potential μ. The macroscopic state of the gas itself is represented by a *thermodynamic potential* specified as a function of thermodynamic parameters, which in the microcanonical ensemble happens to be the internal energy $E_{\text{tot}}(V, S_{\text{tot}}, Q_{\text{tot}})$ of the gas. We can read out various macroscopic observables in terms of the internal energy using the celebrated *first law of thermodynamics*

$$\delta E_{\text{tot}} = T\delta S_{\text{tot}} + \mu\delta Q_{\text{tot}} - P\delta V. \tag{1.1}$$

It states the principle of energy conservation in thermodynamics: the change in the internal energy of a system is equal to the heat $T\delta S_{\text{tot}}$ supplied to the system, plus the energy $\mu\delta Q_{\text{tot}}$ gained by charge influx, minus the work $P\delta V$ done by the system. Based on the application we have in mind, we can also work with any of the other thermodynamic ensembles; they are all related to each other via a Legendre transform. In this work, we are mainly interested in the so-called grand canonical ensemble. It describes a given volume of gas which is free to exchange heat and charge with its surroundings but is held at a fixed temperature and chemical potential. The thermodynamic potential for this ensemble is called the grand potential $\Omega(V, T, \mu)$. It is related to the internal energy of the microcanonical ensemble via the Legendre transform $\Omega = E - TS_{\text{tot}} - \mu Q_{\text{tot}}$. Using the first law of thermodynamics, we can work out its variation to be

$$\delta\Omega = -S_{\text{tot}}\delta T - Q_{\text{tot}}\delta\mu - P\delta V. \tag{1.2}$$

For completeness, we should also mention the third standard statistical ensemble called the canonical ensemble. The associated thermodynamic potential is given by the Helmholtz free energy $F_{\text{tot}}(V, T, Q_{\text{tot}})$ defined as $F_{\text{tot}} = E_{\text{tot}} - TS_{\text{tot}}$. It describes a given volume of gas that is allowed to exchange heat with its surroundings but not charge.

The topic of interest of this work is fluids. From a thermodynamic perspective, fluids are homogeneous systems. That is to say that no particular subvolume of a fluid is more interesting than any generic subvolume of the same fluid. In precise terms, homogeneity can be defined as a property of a thermodynamic system that under a scaling of all the extensive parameters by some arbitrary function λ, the thermodynamic potential also scales by the same factor. For example, in the microcanonical ensemble we have

$$E_{\rm tot}(\lambda V, \lambda S_{\rm tot}, \lambda Q_{\rm tot}) = \lambda E_{\rm tot}(V, S_{\rm tot}, Q_{\rm tot}). \tag{1.3}$$

In other ensembles, this statement equivalently implies $\Omega(\lambda V, T, \mu) = \lambda\Omega(V, T, \mu)$ and $F_{\rm tot}(\lambda V, T, \lambda Q_{\rm tot}) = \lambda F_{\rm tot}(V, T, Q_{\rm tot})$. If we differentiate this equation with respect to λ and set $\lambda = 1$, we arrive at the *Euler equation* for homogeneous fluids

$$E_{\rm tot} = TS_{\rm tot} + \mu Q_{\rm tot} - PV, \tag{1.4}$$

or equivalently $\Omega = -PV$ and $F_{\rm tot} = \mu Q_{\rm tot} - PV$. Note that in the grand canonical ensemble, homogeneity completely fixes the volume dependence of Ω. To see this, note that the pressure of a fluid in the grand canonical ensemble is given by $P = -\partial\Omega/\partial V = -\Omega/V$ which implies $P = P(T, \mu)$ and $\Omega(V, T, \mu) = -P(T, \mu)V$. The functional dependence of P on T and μ, i.e. $P = P(T, \mu)$, is known as the *equation of state* of the fluid, which completely specifies its macroscopic state. It is made manifest by the *Gibbs-Duhem equation*, obtained by substituting the Euler equation into the first law of thermodynamics, leading to

$$\delta P = S\delta T + Q\delta\mu. \tag{1.5}$$

Here we have defined the entropy density $S = S_{\rm tot}/V$ and charge density $Q = Q_{\rm tot}/V$ of the fluid, which are more natural quantities for a homogeneous system. In terms of the densities, the Euler relation itself becomes

$$E + P = TS + \mu Q, \tag{1.6}$$

where $E = E_{\rm tot}/V$ is the internal energy density. Lastly, we can also derive a local version of the first law of thermodynamics

$$\delta E = T\delta S + \mu\delta Q, \tag{1.7}$$

which is more relevant in fluid dynamics. It states that the change in the internal energy distribution of a fluid can be attributed to the change in the heat and charge distributions. Consequently, the equation of state for a fluid in the microcanonical ensemble is $E = E(S, Q)$ and similarly in the canonical ensemble $F = F(T, Q)$, where $F = F_{\rm tot}/V$ is the Helmholtz free energy density.

We should clarify some confusion in the literature regarding the usage of the phrase "equation of state". By our definition, specifying the equation of state completely specifies the macroscopic thermodynamic state of the fluid. This would not be the case if we instead decided to specify, for example, $P(T, Q)$ rather than $P(T, \mu)$. For concreteness, let us look at ideal gases, which are popularly known to have an "equation of state" $PV = Q_{\rm tot}k_{\rm B}T$. In this context, $Q_{\rm tot}$ counts the number of gas atoms/molecules in a given volume and $k_{\rm B}$ is the Boltzmann constant. Dividing by the volume, we get a local version of this equation

$P(T,Q) = Qk_\mathrm{B}T$. Realising that $Q = \partial P/\partial\mu$, we can integrate this equation to get

$$P(T,\mu) = f(T)\exp\left(\frac{\mu}{k_\mathrm{B}T}\right). \tag{1.8}$$

In fact, this would have been the true equation of state of an ideal gas, which, as of yet, is unspecified due to the appearance of an arbitrary function $f(T)$. For instance, we could take this function to be a power law $f(T) = \alpha T^{c_P}$ for some constants α and c_P. Using the Gibbs-Duhem and Euler equations, we can read out the energy, charge, and entropy densities of an ideal gas to be

$$S = \frac{P}{T}\left(c_P - \frac{\mu}{k_\mathrm{B}T}\right), \qquad Q = \frac{P}{k_\mathrm{B}T}, \qquad E = (c_P - 1)P. \tag{1.9}$$

These are, obviously, standard results in the thermodynamics of ideal gases. Importantly, note that the supposed equation of state $PV = Q_\mathrm{tot}k_\mathrm{B}T$ does not contain any information about the specific heat c_P, which however is contained in $P(T,\mu)$.

1.1.2 Relativistic hydrodynamics

Hydrodynamics describes small fluctuations of a homogeneous system around its thermodynamic equilibrium. To this end, we introduce a slowly varying fluid velocity field $u^\mu(t,\mathbf{x})$ normalised as $u^\mu u^\nu \eta_{\mu\nu} = -1$, which in equilibrium would have been just $\delta^\mu{}_t$. Here $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,\ldots)$ is the pseudo-Riemannian Minkowski metric which is used to raise/lower the $\mu,\nu,\ldots$ indices. It should be noted that u^μ characterises a course-grained macroscopic velocity of the fluid at any given spacetime point and not the individual velocities of the microscopic constituents that make up the fluid. The assumption that we can treat fluids as a continuous media, ignoring their microscopic mechanics, is known as the *continuum assumption* of hydrodynamics. Within this premise, we also promote the thermodynamic parameters $T(t,\mathbf{x})$ and $\mu(t,\mathbf{x})$ to slowly varying scalar fields over the spacetime manifold. Here, by “slowly varying” we mean that the spacetime derivatives of u^μ, T, and μ are much smaller compared to the quantities themselves, allowing us to treat derivatives as a well-defined perturbative parameter around the thermodynamic equilibrium. As a corollary, we can always revert back to the thermodynamic regime by setting all the derivatives to zero.

The dynamical setup we have in mind is as follows: let us say that we are given a fluid configuration $u^\mu(0,\mathbf{x})$, $T(0,\mathbf{x})$, and $\mu(0,\mathbf{x})$ at some initial time $t = 0$, along with their first time derivatives. We would like to be able to solve an initial value problem and determine the fluid configuration at all later times $t > 0$. For this purpose, we need a set of equations of motion for u^μ, T, and μ. For the scalars T and μ, we can obtain a first approximation to such an equation by demanding the entropy and charge densities to be conserved along the fluid flow, i.e.

$$\partial_\mu\left(Su^\mu\right) = 0, \qquad \partial_\mu\left(Qu^\mu\right) = 0. \tag{1.10a}$$

Here S and Q are seen as functions of T and μ. The evolution of the fluid velocity, on the other hand, is governed by the relativistic Navier-Stokes equation for force balance

$$(E+P)\, u^\mu \partial_\mu u^\nu = -P^{\nu\rho}\partial_\rho P, \tag{1.10b}$$

where $P^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$ is the projector against the fluid velocity. The fluid acceleration on the left is balanced on the right by the gradient of pressure. The fluids that can be described by this simple set of equations are known as *ideal fluids*.

As the name suggests, the dynamical equations (1.10) are quite idealised. In a generic fluid flow, the flow of entropy might not necessarily align with the flow of charge. Moreover, there could be frictional (viscous) forces in the Navier-Stokes equation due to nearby layers of the fluid dragging along each other. These effects introduce correction terms in eq. (1.10) appearing at second derivative order or higher, which are not fixed by thermodynamics. To efficiently organise these derivative corrections, let us define an *energy-momentum tensor* and *charge current* for our fluid as

$$T^{\mu\nu} = (E+P)u^\mu u^\nu + P\eta^{\mu\nu} + T^{\mu\nu}_{\text{der}}, \qquad J^\mu = Qu^\mu + J^\mu_{\text{der}}. \tag{1.11}$$

Here $T^{\mu\nu}_{\text{der}}$ and J^μ_{der} represent the possible derivative corrections. Using thermodynamics, we can show that the dynamical equations (1.10) can now be represented as the energy-momentum and charge conservation equations

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu J^\mu = 0. \tag{1.12}$$

Thanks to the Noether theorem, these conservation equations are guaranteed to hold even in the presence of derivative corrections. Consequently, derivative corrections only enter the dynamical equations via the corrections in $T^{\mu\nu}$ and J^μ. The plan henceforth is to write down the most generic derivative corrections in the currents $T^{\mu\nu}$ and J^μ allowed by symmetries, truncated to a finite order in the derivative expansion. These expressions are known as the *fluid constitutive relations*.

As such, when we include derivative corrections in $T^{\mu\nu}$ and J^μ, there is no guarantee that the entropy remains conserved. This is not surprising, as we do not expect entropy to stay conserved in an arbitrary dynamical process. However, in accordance with the *second law of thermodynamics*, we still expect there to be a local notion of an entropy current

$$J^\mu_{\text{S}} = Su^\mu + J^\mu_{\text{S,der}}, \tag{1.13}$$

which has a non-negative divergence at every spacetime point, i.e.

$$\partial_\mu J^\mu_{\text{S}} \geq 0. \tag{1.14}$$

This is, in fact, a very non-trivial condition, which needs to be imposed by hand at every derivative order and implies some strict constraints on the tensor structures that can enter the constitutive relations. To glimpse the full potential of this inequality, let us consider the zero derivative order constitutive relations from a combinatorial viewpoint. They are still given by eq. (1.11), except that P, E, and Q are now viewed as arbitrary functions of T and μ, without the thermodynamics relating them. Similarly, the most generic zero order entropy current is given by eq. (1.13). Let us compute the divergence of J^μ_{S}; we find

$$\partial_\mu J^\mu_{\text{S}} = u^\mu \partial_\mu S + S\partial_\mu u^\mu = \left(-\frac{\partial S}{\partial E}(E+P) - \frac{\partial S}{\partial Q}Q + S\right)\partial_\mu u^\mu + \mathcal{O}(\partial^2). \tag{1.15}$$

In the second step we have viewed S as a function of E and Q and used the first order equations of motion

$$\begin{aligned} \partial_\mu T^{\mu\nu} = 0 \implies u^\mu \partial_\mu E &= -(E+P)\partial_\mu u^\mu + u_\nu \partial_\mu T^{\mu\nu}_{\text{der}}, \\ u^\mu \partial_\mu u^\nu &= \frac{1}{E+P}\left(P^{\nu\rho}\partial_\rho P - P^\nu{}_\rho \partial_\mu T^{\mu\rho}_{\text{der}}\right), \\ \partial_\mu J^\mu = 0 \implies u^\mu \partial_\mu Q &= -Q\partial_\mu u^\mu - \partial_\mu J^\mu_{\text{der}}, \end{aligned} \tag{1.16}$$

to eliminate $u^\mu \partial_\mu E$ and $u^\mu \partial_\mu Q$. We want the right-hand side of eq. (1.15) to be non-negative for any fluid profile. Even if we start with a positive $\partial_\mu u^\mu$, there is no guarantee that it remains so during the course of the flow, causing a violation of the second law. The only way of ensuring that the second law is identically upheld is for us to demand the coefficient of $\partial_\mu u^\mu$ in eq. (1.15) to vanish. If at the ideal order, we identify the temperature and chemical potential of the fluid with their thermodynamic values, i.e. $\partial S/\partial E = 1/T + \mathcal{O}(\partial)$ and $\partial S/\partial Q = -\mu/T + \mathcal{O}(\partial)$, we can immediately read out this constraint as the Euler scaling relation. Therefore thermodynamic relations, which are equalities, follow from requiring the inequality of second law.

Let us briefly return to the fluid variables u^μ, T, and μ. At the ideal order, they are uniquely defined by their respective values in thermodynamic equilibrium. However, as we leave the equilibrium, they can admit arbitrary derivative redefinitions

$$T \to T + T_{\text{der}}, \qquad \mu \to \mu + \mu_{\text{der}}, \qquad u^\mu \to u^\mu + u^\mu_{\text{der}}. \tag{1.17}$$

It essentially implies that not all the derivative corrections in $T^{\mu\nu}_{\text{der}}$ and J^μ_{der} are physical; some of them can be attributed to the inherent ambiguity in the fluid variables. Often, it is convenient to work in a *hydrodynamic frame* which uniquely specifies these fluid variables. For instance, in the *Landau frame* one chooses $T^{\mu\nu}u_\nu = -Eu^\mu$ and $J^\mu u_\mu = -Q$, which defines the fluid velocity to be along the flow of energy. There is also the *Eckart frame* with $T^{\mu\nu}u_\mu u_\nu = E$ and $J^\mu = Qu^\mu$, which instead aligns the fluid velocity with the flow of charge. In the bulk of this paper, we work in a third frame called the *hydrostatic frame*. However, we do not have enough tools to define it yet.

As an example, let us consider one-derivative corrections to the fluid constitutive relations. For simplicity, we only include tensor structures that preserve parity. We return to a more general analysis in section 3.1. To start with, let us write down the most generic one-derivative corrections in Landau frame

$$\begin{aligned} T^{\mu\nu}_{\text{der}} &= -\zeta\, P^{\mu\nu}\Theta - \eta\, \sigma^{\mu\nu} + \mathcal{O}(\partial^2), \\ J^\mu_{\text{der}} &= \kappa\, P^{\mu\nu}\partial_\nu T - \sigma\, P^{\mu\nu}\partial_\nu \mu + \mathcal{O}(\partial^2), \end{aligned} \tag{1.18}$$

where we have defined

$$\Theta = \partial_\mu u^\mu, \qquad \sigma^{\mu\nu} = P^{\mu\rho} P^{\nu\sigma} \left(\partial_{(\mu} u_{\nu)} - \frac{1}{3} P_{\mu\nu} \Theta \right). \tag{1.19}$$

We have chosen not to include any terms involving $u^\mu \partial_\mu T$, $u^\mu \partial_\mu \mu$, or $u^\mu \partial_\mu u^\nu$, as they can be eliminated using the first order equations of motion (1.16). Here the bulk viscosity ζ, shear viscosity η, thermal conductivity κ, and electric conductivity σ are arbitrary functions of T and μ, known as *transport coefficients.*

To see what constraints are imposed by the second law of thermodynamics, let us start with the entropy current (1.13) and compute its divergence, but this time paying attention to the derivative corrections. We find

$$\begin{aligned}
\partial_\mu J^\mu_{\rm S} &= -\frac{1}{T} T^{\mu\nu}_{\rm der} \partial_\mu u_\nu - J^\mu_{\rm der} \partial_\mu \frac{\mu}{T} + \partial_\mu \left(J^\mu_{\rm S,der} + \frac{\mu}{T} J^\mu_{\rm der} \right) \\
&= \frac{\zeta}{T} \Theta^2 + \frac{\eta}{T} \sigma^{\mu\nu} \sigma_{\mu\nu} + T\sigma\, P^{\mu\nu} \partial_\mu \frac{\mu}{T} \partial_\nu \frac{\mu}{T} \\
&\quad - \left(\kappa - \frac{\mu}{T} \sigma \right) P^{\mu\nu} \partial_\mu T \partial_\nu \frac{\mu}{T} + \partial_\mu \left(J^\mu_{\rm S,der} + \frac{\mu}{T} J^\mu_{\rm der} \right) + \mathcal{O}(\partial^3).
\end{aligned} \tag{1.20}$$

Here again, we have used the equations of motion to express the entropy current divergence in the desired form. The last term implies that we can choose the derivative corrections in the entropy current to be

$$J^\mu_{\rm S,der} = -\frac{\mu}{T} J^\mu_{\rm der} = -\frac{\mu}{T} \kappa\, P^{\mu\nu} \partial_\nu T + \frac{\mu}{T} \sigma\, P^{\mu\nu} \partial_\nu \mu + \mathcal{O}(\partial^2). \tag{1.21}$$

The remaining term in the last line cannot be made positive definite and hence must vanish. This relates the thermal conductivity to the electric conductivity. On the other hand, the terms in the first line are manifestly positive semi-definite, which tells us that their coefficients must be non-negative, leading to the non-negativity of the viscosities and electric conductivity. Together, we have

$$\kappa = \sigma\mu/T, \qquad \zeta \geq 0, \qquad \eta \geq 0, \qquad \sigma \geq 0. \tag{1.22}$$

In summary, the constitutive relations of a parity-preserving relativistic fluid, corrected up to one-derivative order, are given as

$$\begin{aligned}
T^{\mu\nu} &= (E+P) u^\mu u^\nu + P \eta^{\mu\nu} - \zeta\, P^{\mu\nu} \Theta - \eta\, \sigma^{\mu\nu} + \mathcal{O}(\partial^2), \\
J^\mu &= Q u^\mu - \sigma\, T \partial_\mu \frac{\mu}{T} + \mathcal{O}(\partial^2).
\end{aligned} \tag{1.23}$$

They are characterised by an equation of state $P = P(T, \mu)$ at the ideal order and three non-negative transport coefficients η, ζ, and σ at the first order. These are the standard textbook results and can be found, for example, in [10].

1.1.3 Galilean hydrodynamics

In our discussion above, we assumed the fluid to be relativistic and described its dynamics by a set of Lorentz-invariant dynamical equations. However, for most applications in our day-to-day non-relativistic lives, it is more useful to formulate a Galilean version of hydrodynamics. In a Galilean setting, we still have $T(t,\mathbf{x})$ and $\mu(t,\mathbf{x})$ as fundamental variables of our fluid, but the Galilean fluid velocity is taken to be $u^i(t,\mathbf{x})$ instead, without any normalisation condition. Furthermore, thermodynamics of Galilean fluids can also admit an independent mass density R alongside the charge density, which is absent in a relativistic description. We call the associated chemical potential $\mu_{\mathrm{m}}(t,\mathbf{x})$ with

$$\mathrm{d}P = S\mathrm{d}T + \mu_{\mathrm{m}}\mathrm{d}R + \mu\mathrm{d}Q, \qquad E + P = TS + \mu_{\mathrm{m}}R + \mu Q. \tag{1.24}$$

To first approximation, the dynamical equations for T, μ_{m}, and μ are provided by the entropy, mass, and charge conservation equations respectively, while that for u^i is provided by the Navier-Stokes equation, i.e.

$$\begin{gathered}\partial_t S + \partial_i\left(Su^i\right), \qquad \partial_t R + \partial_i\left(Ru^i\right), \qquad \partial_t Q + \partial_i\left(Qu^i\right), \\ R\left(\partial_t u^i + u^j\partial_j u^i\right) = -\partial^i P.\end{gathered} \tag{1.25}$$

Similar to the relativistic case, the fluids which are governed by this set of equations are known as ideal Galilean fluids.

To classify the possible derivative corrections that these equations can admit, it is useful to convert them into the Noether conservation equations. With this in mind, let us define a set of macroscopic observables for our Galilean fluid

$$\begin{gathered}\rho = R + \rho_{\mathrm{der}}, \qquad \rho^i = Ru^i + \rho^i_{\mathrm{der}}, \\ q = Q + q_{\mathrm{der}}, \qquad j^i = Qu^i + j^i_{\mathrm{der}}, \\ \epsilon = E + \frac{1}{2}Ru^2 + \epsilon_{\mathrm{der}}, \qquad \epsilon^i = \left(E + P + \frac{1}{2}Ru^2\right)u^i + \epsilon^i_{\mathrm{der}}, \\ p^{ij} = Ru^iu^j + P\delta^{ij} + p^{ij}_{\mathrm{der}}.\end{gathered} \tag{1.26}$$

They are the mass density, mass flux, charge density, charge flux, energy density, energy flux, and stress tensor of the fluid respectively. Note that our definition of the energy density ϵ contains contributions from both the internal energy density E as well as the kinetic energy density $\frac{1}{2}Ru^2$, as we would expect for a non-relativistic system. In the expressions above, ρ_{der}, ρ^i_{der}, q_{der}, j^i_{der}, ϵ_{der}, $\epsilon^i_{\mathrm{der}}$, and p^{ij}_{der} represent the possible derivative corrections that the fluid can admit as we leave the thermodynamic equilibrium. In line with our relativistic discussion, we could exploit the redefinition freedom in the hydrodynamic fields to set some of these corrections to zero. One such choice ubiquitous in Galilean hydrodynamics is the so-called *mass frame*, which sets $\rho_{\mathrm{der}} = \rho^i_{\mathrm{der}} = q_{\mathrm{der}} = \epsilon_{\mathrm{der}} = 0$ and aligns the fluid velocity with the flow of mass. Using the definitions in eq. (1.26), we can convert the ideal order equations of motion (1.25) into

$$\partial_t \rho + \partial_i \rho^i = 0, \qquad \partial_t q + \partial_i j^i = 0,$$
$$\partial_t \epsilon + \partial_i \epsilon^i = 0, \qquad \partial_t \rho^i + \partial_j p^{ij} = 0. \tag{1.27}$$

These are the Noether conservation equations for a Galilean system, which we take to be the fundamental equations of motion for Galilean hydrodynamics. Note that we do not require them to admit any explicit derivative corrections; all the corrections enter implicitly via the Galilean currents and densities.

To find these corrections up to any given derivative order, we need to first write down the most generic expressions for various currents and densities allowed by symmetries, called the Galilean fluid constitutive relations. Listing out the tensor structures that respect spatial rotations is quite trivial, as these symmetries are manifest in the index structure of the Galilean observables. However, the Galilean boost symmetry is not manifest; under a boost $x^i \to x^i - \psi^i t$, the fluid velocity shifts as $u^i \to u^i + \psi^i$, and accordingly the Galilean observables mix in a non-trivially manner

$$\rho^i \to \rho^i - \rho\psi^i, \qquad j^i \to j^i - q\psi^i, \qquad p^{ij} \to p^{ij} - 2\rho^{(i}\psi^{j)} + \rho\psi^i\psi^j,$$
$$\epsilon \to \epsilon + \frac{1}{2}\rho\psi^2 - \rho^i\psi_i, \qquad \epsilon^i \to \epsilon^i - \epsilon\psi^i + \frac{1}{2}\psi^2\left(\rho^i - \rho\psi^i\right) - \left(p^{ij} - \psi^i\rho^j\right)\psi_j, \tag{1.28}$$

while leaving the conservation equations invariant. Writing down the tensor structures that are invariant under this transformation can be quite cumbersome. For a detailed account, see the discussion in [10]. However, we can make our life considerably simpler if we introduce an auxiliary coordinate x^- and arrange the Galilean observables into one-higher dimensional Poincaré-invariant structures

$$T^{MN} = \begin{pmatrix} \times & \epsilon & \epsilon^j \\ \epsilon & \rho & \rho^j \\ \epsilon^i & \rho^i & p^{ij} \end{pmatrix}, \qquad J^M = \begin{pmatrix} \times \\ q \\ j^i \end{pmatrix}. \tag{1.29}$$

Here we have chosen the coordinate system $(x^M) = (x^-, t, x^i)$. The coordinates x^- and t are taken to be null with respect to the 5-dimensional spacetime, i.e. $\mathrm{d}s^2 = -2\mathrm{d}x^-\mathrm{d}t + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$. The notation "$\times$" above represents some arbitrary unphysical quantities introduced to complete the 5-dimensional tensor structures; they can be safely ignored for our purposes. Barring these, note that the tensors T^{MN} and J^M are manifestly x^--independent. To ensure that they remain so, we only allow for x^--independent Poincaré transformations on the 5-dimensional spacetime. With some elementary algebra, we can convince ourselves that the residual symmetries are: t and x^i translations, x^i-rotations, U(1) charge transformations, x^- translations, and x^--x^i rotations

$$x^- \to x^- + \frac{1}{2}\psi^2 t - \psi_i x^i, \qquad t \to t, \qquad x^i \to x^i - \psi^i t. \tag{1.30}$$

The spacetime translations, spatial rotations, and U(1) transformations map respectively to their Galilean counterparts, while the x^- translations are identified with an emergent

U(1) symmetry which is responsible for the conservation of mass. The remaining x^--x^i rotations actually map to the non-trivial Galilean boosts, as can be readily verified by applying them to eq. (1.29). We can also arrange the equations of motion (1.27) into a natural 5-dimensional form

$$\partial_M T^{MN} = 0, \qquad \partial_M J^M = 0. \tag{1.31}$$

In deriving these, we have noted that the action of ∂_- yields zero. These equations are precisely the conservation laws of relativistic hydrodynamics given in eq. (1.12). We have, therefore, mapped our Galilean fluids into one-higher dimensional relativistic fluids. Note however that these relativistic fluids are different from those discussed in section 1.1.2; they are anisotropic due the presence of a preferred coordinate x^-. We call them null fluids.

At this point, we can essentially repeat the entire analysis we did for relativistic fluids, step-by-step. For starters, the ideal Galilean fluid constitutive relations (1.26) can be mapped into ideal null fluids as

$$T^{MN} = Ru^M u^N + 2(E+P)u^{(M}V^{N)} + P\eta^{MN} + T^{MN}_{\text{der}}, \qquad J^M = Qu^M + J^M_{\text{der}}, \tag{1.32}$$

where we have defined

$$u^M = \begin{pmatrix} \frac{1}{2}u^2 \\ 1 \\ u^i \end{pmatrix}, \qquad V^M = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \eta^{MN} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \delta^{ij} \end{pmatrix}. \tag{1.33}$$

η^{MN} is just the higher dimensional Minkowski metric written in null coordinates, while V^M is the covariant representation of the null coordinate x^-. On the other hand, u^M is a doubly normalised ($u^M u_M = 0$, $u^M V_M = -1$) null fluid velocity, responsible for the name "null fluids". The tensors T^{MN}_{der} and J^M_{der} represent the possible derivative corrections these constitutive relations can admit. The mass frame condition in this language also takes a natural form: $V_M T^{MN}_{\text{der}} = V_M J^M_{\text{der}} = 0$.

Once we have written down the derivative corrections in accordance with the symmetries, we should impose the second law of thermodynamics. It essentially implies the existence of an entropy density and flux

$$s = S + s_{\text{der}}, \qquad s^i = Su^i + s^i_{\text{der}}, \qquad J^M_{\text{S}} = \begin{pmatrix} \times \\ s \\ s^i \end{pmatrix}, \tag{1.34}$$

whose divergence is positive semi-definite at every spacetime point, i.e.

$$\partial_t s + \partial_i s^i \geq 0 \quad \Longleftrightarrow \quad \partial_M J^M_{\text{S}} \geq 0. \tag{1.35}$$

Similar to the relativistic fluids, this requirement imposes some strict constraints on the Galilean fluid constitutive relations as well. We do not repeat the calculational details here, but following our relativistic steps from eq. (1.15) onward, we can easily infer that the one-derivative corrections to the null fluid constitutive relations are given as

$$
\begin{aligned}
T^{MN} &= Ru^M u^N + 2(E+P)u^{(M}V^{N)} + P\eta^{MN} - \eta\,\sigma^{MN} - \zeta\,P^{MN}\Theta \\
&\quad - 2V^{(M}P^{N)R}\left(\kappa\,\frac{1}{T}\partial_R T + (\kappa_Q + \overline{\kappa}_Q)\,T\partial_R\frac{\mu}{T}\right) + \mathcal{O}(\partial^2), \\
J^M &= Qu^M - (\kappa_Q - \overline{\kappa}_Q)\,P^{MN}\frac{1}{T}\partial_N T - \sigma\,TP^{MN}\partial_N\frac{\mu}{T} + \mathcal{O}(\partial^2).
\end{aligned}
\tag{1.36}
$$

Here we have chosen to work in the mass frame and focused on the parity-preserving sector. We have also defined a projector $P^{MN} = \eta^{MN} + 2u^{(M}V^{N)}$ transverse to u^M and V^M, along with the one-derivative structures

$$
\Theta = \partial_M u^M, \qquad \sigma^{MN} = P^{MR}P^{NS}\left(\partial_{(R}u_{S)} - \frac{1}{3}P_{RS}\Theta\right).
\tag{1.37}
$$

5 of the 6 transport coefficients appearing in eq. (1.36) satisfy a set of 4 inequality constraints among them

$$
\zeta \geq 0, \qquad \eta \geq 0, \qquad \sigma \geq 0, \qquad \kappa \geq \kappa_Q^2/\sigma,
\tag{1.38}
$$

while the remaining transport coefficient $\overline{\kappa}_Q$ is completely arbitrary.

These results can easily be converted to the 4-dimensional language, providing one-derivative corrections to the Galilean fluid constitutive relations in mass frame

$$
\begin{aligned}
j^i &= Qu^i - (\kappa_Q - \overline{\kappa}_Q)\,\frac{1}{T}\partial^i T - \sigma\,T\partial^i\frac{\mu}{T} + \mathcal{O}(\partial^2), \\
\epsilon^i &= \left(E + P + \frac{1}{2}Ru^2\right)u^i - \kappa\,\frac{1}{T}\partial^i T - (\kappa_Q + \overline{\kappa}_Q)\,T\partial^i\frac{\mu}{T} - \eta\,\sigma^{ij}u_j - \zeta\,u^i\Theta + \mathcal{O}(\partial^2), \\
p^{ij} &= Ru^i u^j + P\delta^{ij} - \eta\,\sigma^{ij} - \zeta\,\delta^{ij}\Theta + \mathcal{O}(\partial^2).
\end{aligned}
\tag{1.39}
$$

Written like this, the coefficients can be identified as the bulk viscosity ζ, shear viscosity η, thermal conductivity κ, electric conductivity σ, and two thermo-electric conductivities κ_Q and $\overline{\kappa}_Q$. Note that a Galilean fluid admits many more transport coefficients compared to a relativistic fluid.

This finishes our introductory review of relativistic and Galilean hydrodynamics. We have been quite sketchy in our approach, not spending too much time on the subtleties of the hydrodynamic description. We return to these issues in chapters 2 and 4, where we approach hydrodynamics from a more formal standpoint.

1.2 Hydrodynamics with gapless modes

A major part of this thesis is concerned with how the presence of arbitrary gapless modes in the low-energy spectrum modifies the hydrodynamic framework. So that these results do not get lost in the deluge of technicalities, we outline the basic principles here. For concreteness, we choose the gapless mode in question to be a Goldstone mode φ, arising due to the spontaneous breaking of a U(1) symmetry. Hydrodynamic systems that admit such a Goldstone mode are known as superfluids.

Under a global U(1) transformation, the Goldstone field φ transforms as $\varphi \to \varphi - \Lambda$, so it is more helpful to deal with its derivative $\xi_\mu = \partial_\mu \varphi$ directly, which is U(1)-invariant. The vector field ξ^μ is called the superfluid velocity and is taken to be $\mathcal{O}(\partial^0)$ in the hydrodynamic derivative expansion. Note that the superfluid velocity is exact, i.e. $\partial_{[\mu}\xi_{\nu]} = 0$. The dynamics of φ is governed by the *Josephson equation*

$$u^\mu \xi_\mu = u^\mu \partial_\mu \varphi = \mu + \mu_{\text{der}}, \tag{1.40}$$

where μ_{der} represents the possible derivative corrections that this equation can admit. Although well motivated from a physical standpoint, the Josephson equation is still imposed by hand in the conventional treatment of superfluid dynamics. From a hydrodynamic perspective, where all the dynamics is supposed to emerge on the general grounds of symmetries and thermodynamics, this is quite unnatural. It would be much more natural if we can somehow derive this equation from within the framework of hydrodynamics. More generally, such a derivation will be helpful when we might not have a physical intuition about the low-energy dynamics of a gapless mode. To illustrate how our construction works, let us forget the Josephson equation for now. We will see it emerge as a corollary of the second law of thermodynamics.

The equations of motion for u^μ, T, and μ, on the other hand, are the Noether conservation equations (1.12) as before. However, for our purposes, it is helpful to record a version of these equations which is valid even when φ is taken off-shell[1]

$$\partial_\mu T^{\mu\nu} = K\xi^\nu, \qquad \partial_\mu J^\mu = -K. \tag{1.41}$$

Here we have formally denoted $K = \delta S/\delta\varphi$ with S being an effective action for hydrodynamics. The equation of motion for φ can correspondingly be represented as $K = 0$. If we had started with some other gapless modes than φ, these equations could be considerably different based on the symmetry properties of the gapless modes.

Similar to our previous discussion, we require the fluid to satisfy the second law of thermodynamics. We require the existence of an entropy current J^μ_{S} whose divergence is positive semi-definite everywhere. However, this time we impose this even on arbitrary off-shell configurations of φ. This is quite a non-trivial step, and is responsible for why the second law is able to fix the Josephson equation. To see this, let us work out the ideal superfluid constitutive relations. Using symmetries, at the ideal order the most generic expressions for $T^{\mu\nu}$ and J^μ are given as

$$\begin{aligned} T^{\mu\nu} &= (E+P)u^\mu u^\nu + P\eta^{\mu\nu} + R_s \xi^\mu \xi^\nu + \mathcal{O}(\partial), \\ J^\mu &= Q u^\mu + Q_s \xi^\mu + \mathcal{O}(\partial), \end{aligned} \tag{1.42}$$

where the coefficients are arbitrary functions of T, μ, and $\mu_s = -\frac{1}{2}\xi^\mu\xi_\mu$. On the other

[1] The easiest way to derive these equations is to consider a low energy effective action for φ in thermodynamic equilibrium, i.e. $S[g_{\mu\nu}, A_\mu, \varphi]$, in the presence of arbitrary background sources $g_{\mu\nu}$ and A_μ to couple to the currents $T^{\mu\nu}$ and J^μ respectively. Eq. (1.41) follows from here if we require our action to be invariant under an infinitesimal diffeomorphism: $g_{\mu\nu} \to g_{\mu\nu} + \pounds_\chi g_{\mu\nu}$, $A_\mu \to A_\mu + \pounds_\chi A_\mu$, $\varphi \to \varphi + \pounds_\chi \varphi$ and a U(1) gauge transformation $A_\mu \to A_\mu + \partial_\mu \varphi$, $\varphi \to \varphi - \Lambda$.

hand, K is just an arbitrary scalar which is unfixed for now. We have not included any dependence on $u^\mu\xi_\mu = u^\mu\partial_\mu\varphi$, as this corresponds to the time derivative of φ which can in principle be eliminated using the φ equation of motion. In principle, we could also include a term $2\lambda u^{(\mu}\xi^{\nu)}$ in the energy-momentum tensor, but this gets switched off by the second law of thermodynamics in the end, so we omit it here for the clarity of notation. Using the constitutive relations (1.42), the scalar components of the conservation equations can be evaluated to be

$$\begin{aligned}
\partial_\mu T^{\mu\nu} = K\xi^\nu \implies & u^\mu\partial_\mu E + R_s u^\mu\partial_\mu\mu_s + (E+P)\Theta = -u^\lambda\xi_\lambda\left(K - \partial_\mu\left(R_s\xi^\mu\right)\right), \\
& \zeta^\nu\left((E+P)u^\mu\partial_\mu u_\nu + \partial_\nu P - R_s\partial_\nu\mu_s\right) = \zeta^2\left(K - \partial_\mu\left(R_s\xi^\mu\right)\right), \\
\partial_\mu J^\mu = -K \implies & u^\mu\partial_\mu Q + Q\Theta + \partial_\mu\left(Q_s\xi^\mu\right) = -K,
\end{aligned} \tag{1.43}$$

where $\zeta^\mu = P^{\mu\nu}\xi_\nu$. We use these to eliminate the one-derivative scalars $u^\mu\partial_\mu E$, $\zeta^\nu u^\mu\partial_\mu u_\nu$, and $u^\mu\partial_\mu Q$ respectively. Let us start with an arbitrary zero-derivative entropy current $J^\mu_{\rm S} = Su^\mu + S_s\xi^\mu$ and compute its divergence

$$\begin{aligned}
\partial_\mu J^\mu_{\rm S} &= \frac{\partial S}{\partial E}u^\mu\partial_\mu E + \frac{\partial S}{\partial Q}u^\mu\partial_\mu Q + \frac{\partial S}{\partial\mu_s}u^\mu\partial_\mu\mu_s + S\Theta \\
&= -\left(\frac{\partial S}{\partial E}u^\lambda\xi_\lambda + \frac{\partial S}{\partial Q}\right)\left(K - \partial_\mu\left(R_s\xi^\mu\right)\right) \\
&\quad \left(S_s - (Q_s+R_s)\frac{\partial S}{\partial Q}\right)\partial_\mu\xi^\mu - \frac{\partial S}{\partial Q}\xi^\mu\partial_\mu(Q_s+R_s) + \xi^\mu\partial_\mu S_s \\
&\quad + \left(\frac{\partial S}{\partial\mu_s} - R_s\frac{\partial S}{\partial E}\right)u^\mu\partial_\mu\mu_s + \left(S - (E+P)\frac{\partial S}{\partial E} - Q\frac{\partial S}{\partial Q}\right)\Theta.
\end{aligned} \tag{1.44}$$

We have taken S to be a function of E, Q, and μ_s, and used the first order equations of motion in the second step. Every term in the last two lines of this expression is linearly independent and cannot be made positive semi-definite. Thus all the respective coefficients must vanish, leading to

$$E + P = TS + \mu Q, \qquad R_s = T\frac{\partial S}{\partial\mu_s}, \qquad Q_s = -R_s, \qquad S_s = 0. \tag{1.45}$$

As before, we have identified the temperature and chemical potential with their thermodynamic values: $\partial S/\partial E = 1/T + \mathcal{O}(\partial)$ and $\partial S/\partial Q = -\mu/T + \mathcal{O}(\partial)$. With this, the first condition above can be read as the thermodynamic Euler equation. The first law of thermodynamics and the Gibbs-Duhem equations for a superfluid also follow from the differential expression for S, leading to

$$\mathrm{d}E = T\mathrm{d}S + \mu\mathrm{d}Q - R_s\mathrm{d}\mu_s, \qquad \mathrm{d}P = S\mathrm{d}T + Q\mathrm{d}\mu + R_s\mathrm{d}\mu_s. \tag{1.46}$$

Moving on, we still need to take care of the first term in eq. (1.44). It can be made explicitly positive semi-definite if we choose

$$K = -\frac{\alpha}{T}\left(u^\mu\xi_\mu - \mu\right) + \partial_\mu\left(R_s\xi^\mu\right) + \mathcal{O}(\partial^2), \tag{1.47}$$

for some non-negative transport coefficient α. Using this, the φ equation of motion $K = 0$

takes the form

$$u^\mu \xi_\mu = \mu + \frac{T}{\alpha} \partial_\mu \left(R_s \xi^\mu \right) + \mathcal{O}(\partial). \tag{1.48}$$

This is precisely the Josephson equation. Note that this equation is still only accurate at the zero-derivative order. There can be further one-derivative corrections which we have not analysed here. We return to these in section 3.2.

The framework we have presented here is actually applicable to much more than just superfluids. For a quick example, consider a neutral fluid with an energy momentum tensor $T^{\mu\nu}$. The hydrodynamic fields are u^μ and T, and their dynamics is provided by the energy-momentum conservation equation $\partial_\mu T^{\mu\nu} = 0$. Let us add to this system a dynamical U(1) gauge field A_μ, with field strength $F_{\mu\nu}$, and call the respective equations of motion to be $J^\mu = 0$. Due to the global part of this U(1) gauge symmetry, the system can also admit a chemical potential μ, whose dynamics is provided by $\partial_\mu J^\mu = 0$. In an A_μ-off-shell configuration, the energy-momentum conservation equation modifies to $\partial_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho$. This is essentially the setup for relativistic magnetohydrodynamics [54]. To see this, let us write down the most generic parity-preserving expression for $T^{\mu\nu}$

$$T^{\mu\nu} = (E + P) u^\mu u^\nu + P \eta^{\mu\nu} + \alpha_{\mathrm{BB}} B^\mu B^\nu + \mathcal{O}(\partial), \tag{1.49}$$

where $B^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma}$. All the coefficients are taken to be functions of T, μ, and B^2. Similar to the superfluid case, we have chosen not to include any terms involving $E_\mu = F_{\mu\nu} u^\nu$, as they can be eliminated using the A_μ equations of motion. Performing the same analysis as before, we can read out the thermodynamic relations for magnetohydrodynamics

$$E + P = TS + \mu Q - \alpha_{\mathrm{BB}} B^2, \qquad \mathrm{d}E = T \mathrm{d}S + \mu \mathrm{d}Q - \frac{1}{2} \alpha_{\mathrm{BB}} \mathrm{d}B^2. \tag{1.50}$$

On the other hand, the A_μ equations of motion take the form

$$J^\mu = Q u^\mu - \sigma \left(T \partial^\mu \frac{\mu}{T} - E^\mu \right) - T \partial_\nu \left(\alpha_{\mathrm{BB}} F^{\nu\mu} \right) + \mathcal{O}(\partial^2) = 0, \tag{1.51}$$

for some non-negative transport coefficient σ. These are the finite temperature incarnation of Maxwell's equations. They primarily tell us that on-shell Q and E^μ are $\mathcal{O}(\partial^1)$, which is a characteristic of magnetohydrodynamics. These results can be directly compared to [54].

This is where we close our discussion of gapless modes for now. Later in chapter 2, we revisit the construction of hydrodynamics with arbitrary gapless modes in a more generalised setting. We also briefly mention another application of gapless modes to hydrodynamics with surfaces.

1.3 | Overview and organisation

The material presented in this thesis is quite technical and involved, so we dedicate the remainder of this introductory chapter to chart the essential points. We also take this opportunity to segregate the novel results presented in this thesis from the review material derived from previous works.

The main goal of chapter 2 is to present a universal framework of relativistic hydrodynamics that includes arbitrary gapless modes in its spectrum. Alongside these novel results, the chapter also serves as a comprehensive review of the off-shell formalism of relativistic hydrodynamics, building on the fundamental considerations of symmetries and thermal field theories. We start section 2.1 with a fundamental discussion of symmetries and conserved currents in quantum field theories, motivating their coupling to curved torsional background manifolds. We follow it with a brief review of Einstein-Cartan geometries, giving way to a general discussion on Ward identities and anomalies. Having set the stage, we introduce the formal aspects of relativistic hydrodynamics in section 2.2; we review the essentials of thermal field theory and use it motivate the hydrodynamic fields, their equations of motion, and hydrodynamic constitutive relations. Importantly, we highlight in this section how additional gapless dynamical modes can be dealt with in a hydrodynamic framework, which includes e.g. Goldstone modes of broken internal symmetries for superfluid dynamics and dynamical gauge fields for magnetohydrodynamics. This aspect of chapter 2 is novel to this thesis and has not been published separately. We cap off this section with some comments on the hydrostatic principle and the role of thermal equilibrium in hydrodynamics.

The second law of thermodynamics is at the heart of our modern understanding of hydrodynamics, therefore it gets its own section 2.3. We discuss the hydrodynamic incarnation of the law and explore the classification of hydrodynamic transport it implies. The discussion mainly follows the off-shell formalism of non-Abelian superfluid dynamics we proposed in [1], extended to include spin currents, torsional backgrounds, and arbitrary gapless modes. We find that all the hydrodynamic transport can be classified into one of the following five classes: Class A for anomaly induced transport, Class $\mathrm{H_S}$ and $\mathrm{H_V}$ for hydrostatic transport that governs the behaviour of the fluid in an equilibrium configuration, Class D for dissipative transport that causes the production of entropy when we leave equilibrium, and finally Class $\overline{\mathrm{D}}$ for transport that is neither hydrostatic nor dissipative. There is also a Class S for transport that causes the flow of entropy without any flow of energy-momentum or charge; we mention it for completeness, but it does not play any role in our later discussion. Class $\mathrm{H_S}$ and $\overline{\mathrm{D}}$ are characterised by certain transport coefficients that are arbitrary functions of the thermodynamic variables: temperature and chemical potentials. Class D, as well, is characterised by arbitrary transport coefficients, but specific linear combinations of these are required to be sign-definite by the second law of thermodynamics. On the other hand, Class A and $\mathrm{H_V}$ transport is completely fixed up to some dimensionless constants, called anomaly coefficients and transcendental anomaly coefficients respectively. For Class A, $\mathrm{H_V}$, and $\mathrm{H_S}$ our construction draws heavily from the work of [52], but our characterisation of Class D, $\overline{\mathrm{D}}$, and S is novel.

In chapter 3, we apply the abstract concepts from chapter 2 to some concrete examples. In section 3.1, we revisit the 4-dimensional relativistic fluids from section 1.1.2, but this time from the viewpoint of the off-shell formalism, allowing also for the parity-violating effects. Truncated to one-derivative order, we find a total of 4 constants and 4 transport coefficients classified as $(1_{\mathrm{A}}, 1_{\mathrm{H_S}}, 3_{\mathrm{H_V}}, 3_{\mathrm{D}}, 0_{\overline{\mathrm{D}}})$. The Class $\mathrm{H_S}$ transport coefficient is the ideal order pressure P, while the Class D coefficients are the non-negative shear viscosity η, bulk viscosity ζ, and electric conductivity σ. These results are standard in the literature and have been taken directly from [52]. Next, in section 3.2, we present a generalisation of these results to one-derivative order relativistic superfluids, taken from our work in [4]. We illustrate how the Josephson equation for superfluids naturally emerges from the off-shell formalism. We find that the spectrum of relativistic superfluids is substantially richer than the ordinary fluids, characterised by a total of 2 constants and 31 transport coefficients with classification $(1_{\mathrm{A}}, 5_{\mathrm{H_S}}, 1_{\mathrm{H_V}}, 15_{\mathrm{D}}, 11_{\overline{\mathrm{D}}})$. The 15 dissipative transport coefficients, which include viscosities, conductivities, and their numerous generalisations, satisfy 7 non-trivial inequalities among them. We also find our first non-trivial examples of first-order Class $\mathrm{H_S}$ and $\overline{\mathrm{D}}$ transport coefficients. We also illustrate how these results can be extended to non-Abelian superfluids. In section 3.3, we study the modification of the constitutive relations of a relativistic fluid near a surface and outline a derivation of the Young-Laplace equation. We find that at the one-derivative order the surface dynamics of a fluid is characterised by two additional transport coefficients: an arbitrary Class $\mathrm{H_S}$ surface tension γ and a non-negative Class D transport coefficient α. These results are derived from our work in [6].

In chapter 4, we adapt the off-shell formalism to Galilean hydrodynamics. In a series of collaborative papers [2–5], we have developed a new language to study Galilean fluids, called null fluids, based on the technique of null reduction. These are essentially "relativistic" fluids living on a suitably engineered spacetime manifold, called a null background, with one higher dimension compared to the Galilean fluid of interest. In section 4.1, we start from the Galilean symmetry algebra and review the construction of null backgrounds in detail. Following our work in [5], we discuss the Ward identities for Galilean-invariant field theories and propose a classification of the plausible 't Hooft anomalies. We follow this with a prescription for null reduction in section 4.2, which dimensionally reduces the $(d+1)$-dimensional null background formulation of a Galilean field theory to its conventional d-dimensional formulation. In the d-dimensional picture, we choose to work in the so-called Newton-Cartan framework, which is a covariant language for Galilean field theories coupled to curved spacetime backgrounds and manifests almost all the Galilean symmetries. Alongside, we also include a self-contained review of torsional Newton-Cartan geometries. Finally, in section 4.3, we discuss hydrodynamics on null backgrounds along with its null reduction to obtain Galilean hydrodynamics. We provide a classification scheme implied by the second law of thermodynamics and explore the constraints imposed by it up to all orders in the derivative expansion. Except for the technical details in Class A and $\mathrm{H_V}$, we find the qualitative results to be exactly the same as for the relativistic case. In particular, the second law imposes some strict equality constraints at every derivative-order in the hydrostatic sector, while only requires some inequalities on the one-derivative transport coefficients in the non-hydrostatic sector and none thereafter.

We apply the technique of null fluids to study some examples of Galilean hydrodynamics in chapter 5. In section 5.1, we consider the 4-dimensional ordinary Galilean fluids from section 1.1.3, but in the off-shell formalism. The results have been directly taken from our work in [3]. Including the ideal order pressure, we find a total of 5 constants and 7 transport coefficients classified as $(1_{\mathrm{A}}, 1_{\mathrm{H_S}}, 4_{\mathrm{H_V}}, 5_{\mathrm{D}}, 1_{\overline{\mathrm{D}}})$. There are the shear viscosity η, bulk viscosity ζ, electric conductivity σ, thermal conductivity κ, and thermo-electric conductivity κ_Q in Class D, while there is another thermo-electric conductivity $\overline{\kappa}_Q$ in Class $\overline{\mathrm{D}}$. The inequalities in Class D are $\eta \geq 0$, $\zeta \geq 0$, $\sigma \geq 0$ and $\kappa \geq \kappa_Q^2/\sigma$. Generalising to Galilean superfluids in section 5.2, taken from our work in [4], we find that these numbers skyrocket. We find a total of 2 constants and 52 transport coefficients, classified as $(1_{\mathrm{A}}, 7_{\mathrm{H_S}}, 1_{\mathrm{H_V}}, 25_{\mathrm{D}}, 20_{\overline{\mathrm{D}}})$. The 25 transport coefficients in Class D satisfy 9 inequalities among them. The remaining 7 coefficients in Class $\mathrm{H_S}$ and 20 in Class $\overline{\mathrm{D}}$ are totally unconstrained. In section 5.3, we briefly comment on the surface dynamics in Galilean fluids taken from our work in [6]. Like the relativistic case, we find that the surface transport includes an arbitrary surface tension γ in Class $\mathrm{H_S}$ and a non-negative transport coefficient α in Class D. We also derive the Galilean version of the Young-Laplace equation.

Finally, we close in chapter 6 with some commentary on the results of this thesis and a discussion of possible research directions to be explored in the future.

2 | Relativistic hydrodynamics

In chapter 1 we gave a basic introduction to the principles of relativistic hydrodynamics. We focused on a fluid living on a flat spacetime without any external electromagnetic fields and heuristically derived its constitutive relations up to one-derivative order. In this chapter we deal with issues more formally, accounting for some technicalities which we had swept under the rug in our heuristic introduction. In addition to the spacetime Poincaré symmetry, we permit our fluid to have an internal, possibly non-Abelian, symmetry Lie group G. In the most familiar example of electromagnetically charged fluids this group is just U(1), but we can also consider more exotic groups like the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ of the standard model. The hydrodynamics thus constructed has a direct application to the quark-gluon plasma encountered in accelerator experiments.

A crucial difference compared to most of the existing literature on relativistic hydrodynamics is that we consider a background spacetime with torsion and work in the vielbein formalism. This allows us to describe fluids with an independent "spin current". A secondary motivation for this is our discussion on Galilean fluids in chapter 4, which appears to be mathematically more natural in the presence of torsion. We provide a self-contained introduction to the vielbein formalism and torsion as we go along, but for more details, an excellent review can be found in [85].

2.1 | Preparing the background

2.1.1 Symmetries, currents, and background sources

Symmetries are sacrosanct in physics. Our understanding of the universe is based on, and is very often guided by, the principles of symmetries. In this chapter, we are interested in relativistic quantum field theories which respect spacetime Poincaré symmetries. The corresponding Lie-algebra is generated by

$$\text{Spacetime translations:} \quad P_\alpha, \qquad \text{Lorentz transformations:} \quad J_{\alpha\beta}, \tag{2.1a}$$

with commutation relations

$$\begin{gathered}[P_\alpha, P_\beta] = 0, \qquad [J_{\alpha\beta}, P_\gamma] = i\left(\eta_{\alpha\gamma} P_\beta - \eta_{\beta\gamma} P_\alpha\right), \\ [J_{\alpha\beta}, J_{\gamma\delta}] = i\left(\eta_{\alpha\gamma} J_{\beta\delta} - \eta_{\alpha\delta} J_{\beta\gamma} - \eta_{\beta\gamma} J_{\alpha\delta} + \eta_{\beta\delta} J_{\alpha\gamma}\right).\end{gathered} \tag{2.1b}$$

Here $\eta_{\alpha\beta}$ is the pseudo-Riemannian flat Minkowski metric $\mathrm{diag}(-1, +1, +1, \ldots)$. The indices $\alpha, \beta, \ldots$ run over the d spacetime coordinates. It will be helpful to familiarise oneself with this algebra, as we return to it time and again in this work. The generators $J_{\alpha\beta}$ naturally span a Lorentz algebra $\mathrm{SO}(d{-}1,1)$ which are interpreted as Lorentz-boost and angular momentum operators. The spacetime translation generators P_α are mutually commuting $\mathrm{SO}(d{-}1,1)$ vectors which are seen as spacetime momentum operators. Occasionally, we also permit our theories of interest to admit a Lie group G of internal symmetries, like U(1) for fields charged under electromagnetism or $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ for the full standard model. We call the associated Lie algebra $\mathfrak{g}$, which is endowed with a Lie bracket denoted by $[\circ\,,\circ]$ and a positive semi-definite inner product denoted by $\circ\cdot\circ$, where "$\circ$" is a placeholder for the elements of $\mathfrak{g}$.

Let us say that the theory we are seeking to describe is defined over a d-dimensional spacetime manifold $\mathcal{M}$. We denote the coordinates on $\mathcal{M}$ by the Greek letters $\mu,\nu,\ldots$. Given the symmetries we are working with, Noether's theorem postulates that the spectrum of our theory must contain a set of associated conserved currents

$$\text{Energy-momentum tensor:}\quad T^\mu{}_\alpha\,,\qquad \text{Spin current:}\quad \Sigma^{\mu\alpha}{}_\beta,$$
$$\text{Charge current:}\quad J^\mu. \tag{2.2}$$

The indices $\alpha,\beta,\ldots$ can now be seen as coordinates on a frame bundle $F\mathcal{M}$. To compute the quantum expectation values of these currents in a path-integral formalism, it is convenient to couple the theory to some non-dynamical background sources, one for each current. To achieve this, we introduce on our spacetime manifold $\mathcal{M}$ a set of fields

$$\text{Vielbein:}\quad e^\alpha{}_\mu,\qquad \text{Spin connection:}\quad C^\alpha{}_{\mu\beta}\,,$$
$$\text{Gauge connection:}\quad A_\mu. \tag{2.3}$$

If $\mathcal{Z}[e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu]$ is the field theory generating functional for our theory, which we almost always exchange for $W=-i\ln\mathcal{Z}$, then we can compute the quantum expectation values of various Noether currents via

$$\delta W = \int_{\mathcal{M}} \mathrm{d}^d x\sqrt{-g}\Big(\langle T^\mu{}_\alpha\rangle\delta e^\alpha{}_\mu + \langle\Sigma^{\mu\alpha}{}_\beta\rangle\delta C^\beta{}_{\mu\alpha} + \langle J^\mu\rangle\cdot\delta A_\mu\Big). \tag{2.4}$$

The statement of symmetries can also be made precise in this language, by requiring that the generating functional $W[e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu]$ is invariant under an infinitesimal local diffeomorphism, $\mathfrak{so}(d,1)$ rotation, and $\mathfrak{g}$ transformation (together represented by diff $\times$ $\mathfrak{so}(d{-}1,1)\times\mathfrak{g}$) of the background fields, modulo plausible anomalies. In terms of a set of parameters $\mathcal{X}=(\chi^\mu,\Lambda^{\Sigma\alpha}_\chi{}_\beta,\Lambda_\chi)$, these variations are defined as

$$\begin{aligned}
\delta_{\mathcal{X}} e^\alpha{}_\mu &= \pounds_\chi e^\alpha{}_\mu - \Lambda^{\Sigma\alpha}_\chi{}_\beta\, e^\beta{}_\mu,\\
\delta_{\mathcal{X}} C^\alpha{}_{\mu\beta} &= \pounds_\chi C^\alpha{}_{\mu\beta} + \partial_\mu\Lambda^{\Sigma\alpha}_\chi{}_\beta + [C_\mu,\Lambda^\Sigma_\chi]^\alpha{}_\beta,\\
\delta_{\mathcal{X}} A_\mu &= \pounds_\chi A_\mu + \partial_\mu\Lambda_\chi + [A_\mu,\Lambda_\chi],
\end{aligned} \tag{2.5a}$$

where $\pounds_\chi$ denotes a Lie derivative along χ^μ. These transformations form an algebra: $[\delta_{\mathcal{X}},\delta_{\mathcal{P}}]=\delta_{[\mathcal{X},\mathcal{P}]}$ where $[\mathcal{X},\mathcal{P}]=\delta_{\mathcal{X}}\mathcal{P}=-\delta_{\mathcal{P}}\mathcal{X}$, provided that we define the action of $\mathcal{X}$ on another set of symmetry parameters $\mathcal{P} = (\psi^\mu, \Lambda_\psi^{\Sigma\,\alpha}{}_\beta, \Lambda_\psi)$ as

$$\begin{aligned}
\delta_\mathcal{X}\psi^\mu &= \pounds_\chi \psi^\mu, \\
\delta_\mathcal{X}\Lambda_\psi^{\Sigma\,\alpha}{}_\beta &= \pounds_\chi \Lambda_\psi^{\Sigma\,\alpha}{}_\beta + [\Lambda_\psi^\Sigma, \Lambda_\chi^\Sigma]^\alpha{}_\beta - \pounds_\psi \Lambda_\chi^{\Sigma\,\alpha}{}_\beta \\
\delta_\mathcal{X}\Lambda_\psi &= \pounds_\chi \Lambda_\psi + [\Lambda_\psi, \Lambda_\chi] - \pounds_\psi \Lambda_\chi.
\end{aligned} \tag{2.5b}$$

One might wonder what these transformations have to do with the Poincaré symmetry algebra we originally started with. The simplest way to see the connection is to split the operator $\delta_\mathcal{X}$ in terms of the Poincaré generators as

$$\delta_\mathcal{X} = -i\,\chi^\mu\, e^\alpha{}_\mu P_\alpha - \frac{i}{2}\left(\Lambda_\chi^{\Sigma\,\alpha}{}_\beta + \chi^\mu C^\alpha{}_{\mu\beta}\right) J^\beta{}_\alpha - i\left(\Lambda_\chi + \chi^\mu A_\mu\right)\cdot Q. \tag{2.6}$$

Explicitly computing the commutator $[\delta_\mathcal{X}, \delta_\mathcal{P}]$ in terms of eq. (2.5) and equating it to $\delta_{[\mathcal{X},\mathcal{P}]}$, we can verify that the generators almost satisfy the Poincaré algebra (2.1), except that the spacetime translations no longer commute with each other. Instead, their commutator is sourced by the spacetime torsion, curvature, and field strength associated with the background fields (see the next subsection for the respective definitions) leading to

$$[P_\alpha, P_\beta] = -i\, e_\alpha{}^\mu e_\beta{}^\nu \left(\mathrm{T}^\gamma{}_{\mu\nu} P_\gamma + \frac{1}{2} R_{\mu\nu}{}^\gamma{}_\delta J^\delta{}_\gamma + F_{\mu\nu}\cdot Q\right). \tag{2.7}$$

This also matches our intuitive understanding of the "curved" backgrounds. More importantly, when the background fields are reverted to being flat, for example by setting $e^\alpha{}_\mu = \delta^\alpha{}_\mu$ and $C^\alpha{}_{\mu\beta} = A_\mu = 0$, we recover the full Poincaré algebra (2.1).

2.1.2 Einstein-Cartan geometries

Although not directly relevant as a model of our universe, we see that spacetime geometries with torsion naturally appear as a tool when trying to couple Poincaré invariant quantum field theories with arbitrary background sources. Commonly known as Einstein-Cartan geometries, these were first introduced by Élie Cartan in 1922. We review here some elementary aspects of these geometries which we require in the course of this work. For a more detailed review, please refer to a standard reference like [85]. Let us consider a d-dimensional spacetime manifold $\mathcal{M}$ with a metric $g_{\mu\nu}$ and a generic metric compatible affine connection $\Gamma^\lambda{}_{\mu\sigma}$ given by

$$\Gamma^\lambda{}_{\mu\sigma} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\rho\sigma} + \partial_\sigma g_{\rho\mu} - \partial_\rho g_{\mu\sigma} + \mathrm{T}_{\rho\mu\sigma} - \mathrm{T}_{\mu\sigma\rho} - \mathrm{T}_{\sigma\mu\rho}\right). \tag{2.8}$$

The antisymmetric part of the affine connection $\mathrm{T}^\lambda{}_{\mu\nu} = -\mathrm{T}^\lambda{}_{\nu\mu}$ is called the Cartan torsion tensor. We also define on $\mathcal{M}$ a $\mathfrak{g}$-valued gauge field A_μ, and take D_μ to be the covariant derivative operator associated with $\Gamma^\lambda{}_{\mu\sigma}$ and A_μ. We can work out the Riemann curvature tensor and gauge field strength associated with these connections as

$$R_{\mu\nu}{}^\lambda{}_\sigma = 2\partial_{[\mu}\Gamma^\lambda{}_{\nu]\sigma} + 2\Gamma^\lambda{}_{[\mu|\rho}\Gamma^\rho{}_{\nu]\sigma}, \qquad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu]. \tag{2.9}$$

Physical theories living on $\mathcal{M}$ are required to respect spacetime diffeomorphisms and internal G-transformations. Let us denote an infinitesimal such transformation by parameters $\mathcal{X} = (\chi^\mu, \Lambda_\chi)$, where χ^μ is a vector field and Λ_χ is a $\mathfrak{g}$-valued scalar field. The derivative operator D_μ acts on these fields as

$$\mathrm{D}_\mu \chi^\nu = \partial_\mu \chi^\nu + \Gamma^\nu{}_{\mu\rho}\chi^\rho, \qquad \mathrm{D}_\mu \Lambda_\chi = \partial_\mu \Lambda_\chi + [A_\mu, \Lambda_\chi]. \tag{2.10}$$

Under the action of $\mathcal{X}$, the background fields transform as

$$\begin{aligned}
\delta_{\mathcal{X}} g_{\mu\nu} &= \pounds_\chi g_{\mu\nu} &&= 2\mathrm{D}_{(\mu}\chi_{\nu)} - \mathrm{T}_{(\mu\nu)\sigma}\chi^\sigma, \\
\delta_{\mathcal{X}} \Gamma^\lambda{}_{\mu\sigma} &= \pounds_\chi \Gamma^\lambda{}_{\mu\sigma} + \partial_\mu\partial_\sigma\chi^\lambda &&= \mathrm{D}_\mu\left(\mathrm{D}_\sigma\chi^\lambda + \chi^\nu \mathrm{T}^\lambda{}_{\nu\sigma}\right) + \chi^\nu R_{\nu\mu}{}^\lambda{}_\sigma, \\
\delta_{\mathcal{X}} A_\mu &= \pounds_{\mathcal{X}} A_\mu + \mathrm{D}_\mu\Lambda_\chi &&= \mathrm{D}_\mu\left(\Lambda_\chi + \chi^\nu A_\nu\right) + \chi^\nu F_{\nu\mu}.
\end{aligned} \tag{2.11}$$

To this end, our background geometry is described by a spacetime manifold $\mathcal{M}$ with fields $(g_{\mu\nu}, \Gamma^\lambda{}_{\mu\sigma}, A_\mu)$ modded by local diff $\times\, \mathfrak{g}$ transformations.

In the case of torsional geometries, however, it is more natural to shift to the vielbein formalism, which we describe in the following. The condition of local flatness of a manifold allows us to define an isomorphism between the tangent bundle $\mathrm{T}\mathcal{M}$ and a pseudo-Riemannian frame bundle $F\mathcal{M} = \mathbb{R}^{(d-1,1)}$, realised in terms of a vielbein $e^\alpha{}_\mu$ and its inverse $e_\alpha{}^\mu$. The vielbein is essentially a d^2 component matrix, defined by the requirement that it maps the metric $g_{\mu\nu}$ on $\mathcal{M}$ to a flat Minkowski metric $\eta_{\alpha\beta}$ on $F\mathcal{M}$,

$$g_{\mu\nu} e_\alpha{}^\mu e_\beta{}^\nu = \eta_{\alpha\beta}, \qquad g^{\mu\nu} e^\alpha{}_\mu e^\beta{}_\nu = \eta^{\alpha\beta}. \tag{2.12}$$

We often use the vielbein to freely switch between the spacetime indices $\mu, \nu, \ldots$ and the flat indices $\alpha, \beta, \ldots$. Note that the defining equation (2.12) for a vielbein has $\frac{1}{2}d(d+1)$ components, so it leaves $\frac{1}{2}d(d-1)$ of its components undetermined. This redundancy can be attributed to a $\mathrm{SO}(d{-}1,1)$ Lorentz symmetry on $F\mathcal{M}$ that acts on the vielbein as $e^\alpha{}_\mu \to O^\alpha{}_\beta e^\beta{}_\mu$ where $O^\alpha{}_\gamma O^\beta{}_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta}$. It is trivial to check that eq. (2.12) is invariant under this transformation. We can define a spin-connection for fields transforming in some non-trivial representation of this symmetry

$$C^\alpha{}_{\mu\beta} \equiv e_\beta{}^\sigma\left(e^\alpha{}_\rho \Gamma^\rho{}_{\mu\sigma} - \partial_\mu e^\alpha{}_\sigma\right), \tag{2.13}$$

which has the same amount of information as $\Gamma^\lambda{}_{\mu\sigma}$. One can check that with this connection, $\mathrm{D}_\mu e^a{}_\nu = 0$, which is tantamount to the metric compatibility of $\Gamma^\lambda{}_{\mu\sigma}$. Switching momentarily to differential forms notation we define

$$\boldsymbol{e}^\alpha = e^\alpha{}_\mu \mathrm{d}x^\mu, \qquad \boldsymbol{C}^\alpha{}_\beta = C^\alpha{}_{\mu\beta}\mathrm{d}x^\mu, \qquad \boldsymbol{A} = A_\mu \mathrm{d}x^\mu, \tag{2.14}$$

in terms of which the torsion, curvature, and field strength can be expressed as

$$\begin{aligned}
\boldsymbol{T}^\alpha &= \frac{1}{2}\mathrm{T}^\alpha{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}\boldsymbol{e}^\alpha + \boldsymbol{C}^\alpha{}_\beta \wedge \boldsymbol{e}^\beta = \Big(\partial_{[\mu}e^\alpha{}_{\nu]} + C^\alpha{}_{[\mu\beta}e^\beta{}_{\nu]}\Big)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= \frac{1}{2}e^\alpha{}_\lambda \mathrm{T}^\lambda{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \\
\boldsymbol{R}^\alpha{}_\beta &= \frac{1}{2}R_{\mu\nu}{}^\alpha{}_\beta \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}\boldsymbol{C}^\alpha{}_\beta + \boldsymbol{C}^\alpha{}_\gamma \wedge \boldsymbol{C}^\gamma{}_\beta = \big(\partial_{[\mu}C^\alpha{}_{\nu]\beta} + C^\alpha{}_{[\mu\gamma}C^\gamma{}_{\nu]\beta}\big)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= \frac{1}{2}e^\alpha{}_\lambda e_\beta{}^\sigma R_{\mu\nu}{}^\lambda{}_\sigma \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \\
\boldsymbol{F} &= \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}\boldsymbol{A} + \boldsymbol{A} \wedge \boldsymbol{A}.
\end{aligned} \tag{2.15}$$

Under an infinitesimal local diff $\times\, \mathfrak{so}(d{-}1,1) \times \mathfrak{g}$ transformation parametrised by $\mathcal{X} = (\chi^\mu, \Lambda^{\Sigma\alpha}_\chi{}_\beta, \Lambda_\chi)$ various background fields vary according to

$$\begin{aligned}
\delta_\mathcal{X} e^\alpha{}_\mu &= \pounds_\chi e^\alpha{}_\mu - \Lambda^{\Sigma\alpha}_\chi{}_\beta\, e^\beta{}_\mu &&= e^\alpha{}_\nu \mathrm{D}_\mu \chi^\nu + \chi^\nu \mathrm{T}^\alpha{}_{\nu\mu} - \big(\Lambda^{\Sigma\alpha}_\chi{}_\beta + \chi^\nu C^\alpha{}_{\nu\beta}\big)\, e^\beta{}_\mu, \\
\delta_\mathcal{X} C^\alpha{}_{\mu\beta} &= \pounds_\xi C^\alpha{}_{\mu\beta} + \mathrm{D}_\mu \Lambda^{\Sigma\alpha}_\chi{}_\chi &&= \mathrm{D}_\mu\left(\Lambda^{\Sigma\alpha}_\chi{}_\beta + \chi^\nu C^\alpha{}_{\nu\beta}\right) + \chi^\nu R_{\nu\mu}{}^\alpha{}_\beta, \\
\delta_\mathcal{X} A_\mu &= \pounds_\chi A_\mu + \mathrm{D}_\mu \Lambda_\chi &&= \mathrm{D}_\mu\left(\Lambda_\chi + \chi^\nu A_\nu\right) + \chi^\nu F_{\nu\mu}.
\end{aligned} \tag{2.16}$$

These follow trivially from the respective definitions of various quantities and are exactly the same as advertised in eq. (2.5). In the vielbein formalism, therefore, our background geometry is a d-dimensional spacetime manifold $\mathcal{M}$ with fields $(e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu)$, but this time modded by diff $\times\, \mathfrak{so}(d{-}1,1) \times \mathfrak{g}$ transformations.

This is the extent of the Einstein-Cartan geometries that we require in this chapter. There are, however, some interesting identities and relations which might be useful in an explicit computation. We refer the reader to [85] for a detailed review.

2.1.3 Noether theorem and Ward identities

Now that we have our background and symmetries ready, in this subsection we briefly outline the characteristic features of the physical theories that are coupled to it. Let the theory in question be described by some dynamical fields φ^I with respective equations of motion $\mathcal{E}_I \approx 0$.[1] We assume for now that our theory is described by an action $S[e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu; \varphi^I]$. Since we are dealing with low energy effective descriptions in this work, we do not require the theory to be UV complete. This essentially means that φ^I could be some effective degrees of freedom relevant at our energy scale and S could be a Wilsonian effective action, with all the heavier modes integrated out.

Under an infinitesimal variation of the background and dynamical fields, we can parametrise the variation of the action as

$$\delta S = \int \mathrm{d}^d x \sqrt{-g}\Big(T^\mu{}_\alpha\, \delta e^\alpha{}_\mu + \Sigma^{\mu\alpha}{}_\beta\, \delta C^\beta{}_{\mu\alpha} + J^\mu \cdot \delta A_\mu + \mathcal{E}_I\, \delta\varphi^I\Big). \tag{2.17}$$

[1] Throughout this work we denote on-shell equalities by "≈", whereas we keep "=" reserved for off-shell statements. Later we also introduce the symbol "≃" for "partially on-shell" statements, where only the hydrodynamic fields have been taken on-shell.

Here we have identified the Noether currents as being probed by the background fields. The variation of the action with respect to φ^I, on the other hand, gives us the equations of motion $\mathcal{E}_I$. Since the theory we seek to describe is invariant under the action of a set of symmetry parameters $\mathcal{X}$, we must require $\delta_{\mathcal{X}} S = 0$, modulo plausible anomalies. We already know how $\mathcal{X}$ acts on the background fields, but we cannot say anything about the dynamical fields φ^I without knowing their explicit transformation properties. Nevertheless, provided that the symmetries act homogeneously on these fields, using differentiation by parts we can write down a generic statement

$$\mathcal{E}^I \delta_{\mathcal{X}} \varphi_I = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, \mathcal{N}^{I\mu}_{\mathcal{X}} \mathcal{E}_I \right) + \left[\chi^\alpha \mathcal{O}^I_\alpha + \left(\Lambda_\chi^{\Sigma\alpha}{}_\beta + \chi^\nu C^\alpha{}_{\nu\beta} \right) \mathcal{O}^{I\beta}{}_\alpha + \left(\Lambda_\chi + \chi^\mu A_\mu \right) \cdot \mathcal{O}^I \right] \mathcal{E}_I. \quad (2.18)$$

Here $\mathcal{O}^I$'s are some $\mathcal{X}$-independent differential operators, while the operator $\mathcal{N}^{I\mu}_{\mathcal{X}}$ is linear in $\mathcal{X}$. The explicit form of these operators depends on how $\mathcal{X}$ acts on φ^I and can be left abstract for the purposes of this generic discussion. Plugging the field variations from eqs. (2.16) and (2.18) into eq. (2.17) and requiring $\delta_{\mathcal{X}} S$ to vanish modulo anomalies, we can read out a set of identities

$$\begin{aligned}
\underline{\mathrm{D}}_\mu T^\mu{}_\alpha &= e_\alpha{}^\nu \left(\mathrm{T}^\beta{}_{\nu\mu} T^\mu{}_\beta + R_{\nu\mu}{}^\gamma{}_\beta \Sigma^{\mu\beta}{}_\gamma + F_{\nu\mu} \cdot J^\mu \right) + \mathcal{O}^I_\alpha \mathcal{E}_I, \\
\underline{\mathrm{D}}_\mu \Sigma^{\mu\alpha\beta} &= T^{[\beta\alpha]} + \Sigma_{\mathrm{H}}^{\perp\alpha\beta} + \mathcal{O}^{I\alpha\beta} \mathcal{E}_I, \\
\underline{\mathrm{D}}_\mu J^\mu &= \mathrm{J}^\perp_{\mathrm{H}} + \mathcal{O}^I \mathcal{E}_I.
\end{aligned} \quad (2.19)$$

We have defined the notation $\underline{\mathrm{D}}_\mu = \mathrm{D}_\mu + e_\alpha{}^\nu \mathrm{T}^\alpha{}_{\mu\nu}$ to avoid clutter. The Hall currents $\Sigma_{\mathrm{H}}^{\perp\alpha}{}_\beta$ and $\mathrm{J}^\perp_{\mathrm{H}}$ are a manifestation of anomalies, which we discuss in detail in section 2.1.4. For now it suffices to say that they are completely fixed in terms of $R_{\mu\nu}{}^\alpha{}_\beta$ and $F_{\mu\nu}$ up to some constants. On-shell, when $\mathcal{E}_I \approx 0$, the Ward identities (2.19) imply a set of conservation laws for $T^\mu{}_\alpha$, $\Sigma^{\mu\alpha}{}_\beta$, and J^μ respectively

$$\begin{aligned}
\underline{\mathrm{D}}_\mu T^\mu{}_\alpha &\approx e_\alpha{}^\nu \left(\mathrm{T}^\beta{}_{\nu\mu} T^\mu{}_\beta + R_{\nu\mu}{}^\gamma{}_\beta \Sigma^{\mu\beta}{}_\gamma + F_{\nu\mu} \cdot J^\mu \right), \\
\underline{\mathrm{D}}_\mu \Sigma^{\mu\alpha\beta} &\approx T^{[\beta\alpha]} + \Sigma_{\mathrm{H}}^{\perp\alpha\beta}, \\
\underline{\mathrm{D}}_\mu J^\mu &\approx \mathrm{J}^\perp_{\mathrm{H}}.
\end{aligned} \quad (2.20)$$

These are the Noether conservation laws corresponding to our symmetries, which along with the identities eq. (2.19), form the backbone of all our analysis in this work.

When the background fields are switched off, i.e. $e^\alpha{}_\mu = \delta^\alpha{}_\mu$ and $C^\alpha{}_{\mu b} = A_\mu = 0$, the conservation laws reduce to their better known form

$$\partial_\mu T^\mu{}_\alpha \approx 0, \qquad \partial_\mu \Sigma^{\mu\alpha\beta} \approx T^{[\beta\alpha]}, \qquad \partial_\mu J^\mu \approx 0. \quad (2.21)$$

2.1.4 Anomalous symmetries

In our discussion above, we refereed to the so-called Hall currents $\Sigma_{\mathrm{H}}^{\perp\alpha\beta}$ and $\mathrm{J}_{\mathrm{H}}^{\perp}$, which we claimed were a manifestation of our symmetries being anomalous. Let us take a brief detour and discuss what these currents actually are and how they arise using the anomaly inflow mechanism. In relativistic field theories, the anomaly inflow mechanism is an efficient way to classify flavour and gravitational/Lorentz anomalies [86]. A detailed introductory discussion can be found in section 2 of [87]. In essence, we consider that our d-dimensional spacetime manifold $\mathcal{M}$ lives on the boundary of a $(d+1)$-dimensional *bulk* manifold $\mathcal{B}$. We denote the bulk coordinates with a hat and near the boundary choose a basis $(x^{\hat{\mu}}) = (x^{\perp}, x^{\mu})$, where $x^{\perp}$ corresponds to the depth into the bulk. All the field content $e^{\hat{\alpha}}{}_{\hat{\mu}}$, $C^{\hat{\alpha}}{}_{\hat{\mu}\hat{\beta}}$ and $A_{\hat{\mu}}$ is extended down into the bulk with the requirement that all the $x^{\perp}$ components vanish at the boundary.

We keep our theory of interest on $\mathcal{M}$, whose field theory generating functional $W_{\mathcal{M}}$ is not necessarily invariant under the symmetries of the theory, i.e. is anomalous. In the bulk we keep some topological theory with generating functional $W_{\mathcal{B}}$, which is invariant under all the symmetries up to some non-trivial boundary terms. The full theory described by $W = W_{\mathcal{M}} + W_{\mathcal{B}}$ is assumed to be invariant under all the symmetries. It is actually this non-trivial bulk term $W_{\mathcal{B}}$ which induces anomaly in the boundary theory, hence the name *anomaly inflow*. In the notation of differential forms, $W_{\mathcal{B}}$ can be expressed as an integration of a $(d+1)$-rank form $\boldsymbol{I}$,

$$W_{\mathcal{B}} = \int_{\mathcal{B}} \boldsymbol{I}. \tag{2.22}$$

The requirement that $W_{\mathcal{B}}$ should be symmetry-invariant up to a boundary term can be recast into the requirement that $\boldsymbol{\mathcal{P}} = \mathrm{d}\boldsymbol{I}$ should be invariant under all the symmetries. $\boldsymbol{\mathcal{P}}$ is called the *anomaly polynomial* of the theory, which encodes all the non-trivial information about anomaly. It is evident that $\boldsymbol{\mathcal{P}}$ needs to be closed, symmetry invariant, and should not be expressible as exterior derivative of a symmetry invariant form. Its explicit form, however, depends on the background field content of the theory. In the current context, it is given by the Chern-Simons anomaly polynomial $\boldsymbol{\mathcal{P}}_{\mathrm{CS}}$ for even dimensional boundary theories, while no such terms are possible in odd spacetime dimensions. Here $\boldsymbol{\mathcal{P}}_{\mathrm{CS}}$ is a "polynomial" made out of the Chern classes of the field strength $\boldsymbol{F}$ and the Pontryagin classes of the curvature tensor $\boldsymbol{R}^{\hat{\alpha}}{}_{\hat{\beta}}$. In $d = 4$, for example, the most generic such anomaly polynomial is given as

$$\boldsymbol{\mathcal{P}}_{\mathrm{CS}} = C\ \mathrm{tr}[\boldsymbol{F}\wedge\boldsymbol{F}\wedge\boldsymbol{F}] + C_g\ \mathrm{tr}\,\boldsymbol{F}\wedge\boldsymbol{R}^{\hat{\alpha}}{}_{\hat{\beta}}\wedge\boldsymbol{R}^{\hat{\beta}}{}_{\hat{\alpha}}, \tag{2.23}$$

where the trace is defined over the adjoint representation of the Lie-algebra $\mathfrak{g}$. The constants C and C_g are called the anomaly coefficients. Look at e.g. [87] for more details. In generality, for $d = 4k-2$ or $d = 4k$, there are $k+1$ possible anomaly coefficients.

Let us parametrise an infinitesimal variation of the bulk generating functional $W_{\mathcal{B}}$ as

$$\delta W_{\mathcal{B}} = \int_{\mathcal{B}} \mathrm{d}^{d+1}x\sqrt{-g_{d+1}}\left(\Sigma_{\mathrm{H}}{}^{\hat\mu\hat\alpha}{}_{\hat\beta}\delta C^{\hat\beta}{}_{\hat\mu\hat\alpha} + \mathrm{J}^{\hat\mu}_{\mathrm{H}}\cdot\delta A_{\hat\mu}\right) \\ + \int_{\mathcal{M}} \mathrm{d}^d x\sqrt{-g}\left(\Sigma_{\mathrm{BZ}}{}^{\mu\alpha}{}_{\beta}\delta C^{\beta}{}_{\mu\alpha} + \mathrm{J}^{\mu}_{\mathrm{BZ}}\cdot\delta A_{\mu}\right). \quad (2.24)$$

The currents in the bulk subscripted by "H" are called *Hall currents*. In terms of the anomaly polynomial, they can be formally written down as a compact formula[2]

$$\star_{(d+1)}\boldsymbol{\Sigma}_{\mathrm{H}}{}^{\hat\alpha}{}_{\hat\beta} = \frac{\partial\boldsymbol{\mathcal{P}}}{\partial\boldsymbol{R}^{\hat\beta}{}_{\hat\alpha}}, \qquad \star_{(d+1)}\mathbf{J}_{\mathrm{H}} = \frac{\partial\boldsymbol{\mathcal{P}}}{\partial\boldsymbol{F}}. \quad (2.25)$$

Here the Hodge duality operator $\star_{(d+1)}$ is defined in the bulk. For our $d=4$ example in eq. (2.23), we can explicitly compute the Hall currents to be

$$\Sigma_{\mathrm{H}}{}^{\hat\mu\hat\alpha}{}_{\hat\beta} = \frac{1}{2}C_g\,\epsilon^{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau}\,\mathrm{tr}\,F_{\hat\nu\hat\rho}R_{\hat\sigma\hat\tau}{}^{\hat\alpha}{}_{\hat\beta}, \\ \mathrm{J}^{\hat\mu}_{\mathrm{H}} = \frac{3}{4}C\,\epsilon^{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau}F_{\hat\nu\hat\rho}F_{\hat\sigma\hat\tau} + \frac{1}{4}C_g\,\epsilon^{\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau}R_{\hat\nu\hat\rho}{}^{\hat\alpha}{}_{\hat\beta}R_{\hat\sigma\hat\tau}{}^{\hat\beta}{}_{\hat\alpha}. \quad (2.26)$$

The term coupling to C is called G (flavour) anomaly, because it only affects the G-symmetry sector, while C_g is called a mixed anomaly as it affects both Lorentz as well as G sectors.

On the other hand, the bulk generating functional induces the so called *Bardeen-Zumino* currents subscripted by "BZ" at the boundary. They can be represented as derivatives of $\boldsymbol{I}$, i.e.

$$\star\boldsymbol{\Sigma}_{\mathrm{BZ}}{}^{\hat\alpha}{}_{\hat\beta} = \frac{\partial\boldsymbol{I}}{\partial\boldsymbol{R}^{\hat\beta}{}_{\hat\alpha}}, \qquad \star\mathbf{J}_{\mathrm{BZ}} = \frac{\partial\boldsymbol{I}}{\partial\boldsymbol{F}}. \quad (2.27)$$

These are, of course, not symmetry-covariant. But since the full partition function W is symmetry-covariant, if we define the variation of the boundary piece $W_{\mathcal{M}}$ as

$$\delta W_{\mathcal{M}} = \int_{\mathcal{M}} \mathrm{d}^d x\sqrt{-g}\left(T^{\mu}{}_{\alpha}\,\delta e^{\alpha}{}_{\mu} + \Sigma_{\mathrm{cons}}{}^{\mu\alpha}{}_{\beta}\,\delta C^{\beta}{}_{\mu\alpha} + J^{\mu}_{\mathrm{cons}}\cdot\delta A_{\mu}\right), \quad (2.28)$$

the full boundary *covariant currents*

$$T^{\mu}{}_{\alpha}, \qquad \Sigma^{\mu\alpha}{}_{\beta} = \Sigma_{\mathrm{cons}}{}^{\mu\alpha}{}_{\beta} + \Sigma_{\mathrm{BZ}}{}^{\mu\alpha}{}_{\beta}, \qquad J^{\mu} = J^{\mu}_{\mathrm{cons}} + \mathrm{J}^{\mu}_{\mathrm{BZ}}, \quad (2.29)$$

are symmetry-covariant. Moreover, demanding $\delta_{\mathfrak{X}}W = 0$ under an infinitesimal symmetry variation $\mathfrak{X}$ precisely leads to the conservation laws (2.20) for the boundary currents. There are also a similar set of (non-anomalous) conservation laws in the bulk, but given that there are no dynamical fields in the bulk, they are trivially satisfied.

As opposed to the covariant boundary currents, the currents obtained from $W_{\mathcal{M}}$, i.e. $(T^{\mu}{}_{\alpha}, \Sigma_{\mathrm{cons}}{}^{\mu\alpha}{}_{\beta}, J^{\mu}_{\mathrm{cons}})$, are called the *consistent currents*. Even though they are non-covariant under symmetries, they are the physical Noether currents of the boundary theory. They

[2]We should clarify what we mean by differentiating or dividing with respect to a 2-form. Since $\boldsymbol{\mathcal{P}}$ is a polynomial in 2-forms, which mutually commute, we can intuitively define this differentiation and division as we would for an ordinary polynomial. Such an operation converts a $(d+2)$-form into a d-form.

also satisfy a set of conservation laws akin to eq. (2.20),

$$\begin{aligned}
\underline{\mathrm{D}}_\mu T^\mu{}_\alpha &\approx e_\alpha{}^\nu \left(\mathrm{T}^\beta{}_{\nu\mu} T^\mu{}_\beta + R_{\nu\mu}{}^\gamma{}_\beta \Sigma_{\mathrm{cons}}{}^{\mu\beta}{}_\gamma + F_{\nu\mu} \cdot J^\mu_{\mathrm{cons}} \right) + \mathrm{T}_{\mathrm{H,cons}}{}^\perp{}_\alpha, \\
\underline{\mathrm{D}}_\mu \Sigma^{\mu\alpha\beta}_{\mathrm{cons}} &\approx T^{[\beta\alpha]} + \Sigma^{\perp\alpha\beta}_{\mathrm{H,cons}}, \\
\underline{\mathrm{D}}_\mu J^\mu_{\mathrm{cons}} &\approx \mathrm{J}^\perp_{\mathrm{H,cons}},
\end{aligned} \tag{2.30}$$

but the “consistent anomaly” is instead given as

$$\begin{gathered}
\mathrm{T}_{\mathrm{H,cons}}{}^\perp{}_\alpha = e_\alpha{}^\nu \left(R_{\nu\mu}{}^\gamma{}_\beta \Sigma_{\mathrm{BZ}}{}^{\mu\beta}{}_\gamma + F_{\nu\mu} \cdot \mathrm{J}^\mu_{\mathrm{BZ}} \right), \\
\Sigma_{\mathrm{H,cons}}{}^{\perp\alpha}{}_\beta = \Sigma^{\perp\alpha}_{\mathrm{H}}{}_\beta - \underline{\mathrm{D}}_\mu \Sigma_{\mathrm{BZ}}{}^{\mu\alpha}{}_\beta, \qquad \mathrm{J}^\perp_{\mathrm{H,cons}} = \mathrm{J}^\perp_{\mathrm{H}} - \underline{\mathrm{D}}_\mu \mathrm{J}^\mu_{\mathrm{BZ}}.
\end{gathered} \tag{2.31}$$

In particular, note that written in terms of the consistent currents, the energy-momentum conservation also gets an anomaly. In most of this work, however, we are working directly with the covariant currents.

2.2 The hydrodynamic setup

We are interested in a universal low energy effective description of finite-temperature near-equilibrium field theories. By itself, it is an absurd thing to seek as it would obviously depend on the field theory under consideration. But in most physical scenarios when such an effective description would actually be useful, like describing your mug of coffee as you add milk to it, we probably would not know anything about the underlying microscopic field theory, let alone an action principle. Within its narrow regime of applicability, hydrodynamics aims at providing a universal framework to study low energy fluctuations of a finite temperature field theory around its thermodynamic equilibrium with unknown microscopics, guided by the fundamental principles like symmetries and other empirical physical expectations like the second law of thermodynamics.

2.2.1 Thermal equilibrium

Before considering departures from equilibrium, it will be helpful to revisit some elementary facts about quantum field theories at thermodynamic equilibrium. The material presented in this section can be found in any standard text on thermal field theories like [88, 89]. Consider a relativistic background manifold $\mathcal{M}$ constructed in the previous section with fields $(e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu)$. $\mathcal{M}$ is said to admit an equilibrium if there exists a set of symmetry parameters $\mathcal{K} = (K^\mu, \Lambda^\Sigma_K, \Lambda_K)$ with $K_\mu K^\mu < 0$, whose action on $\mathcal{M}$ is an isometry

$$\delta_{\mathcal{K}} e^\alpha{}_\mu = \delta_{\mathcal{K}} C^\alpha{}_{\mu\beta} = \delta_{\mathcal{K}} A_\mu = 0. \tag{2.32}$$

$\mathcal{K}$ can be interpreted as a set of parameters defining a relativistic observer, with respect to whom the background is time-independent. In $\mathcal{K}$'s reference frame, the time-evolution of an observable is generated by a time-independent *Hamiltonian* operator $\mathcal{H}_{\mathcal{K}}$ given by

$$\mathcal{H}_{\mathcal{K}} = \int_\Sigma \mathrm{d}^{d-1}x\sqrt{-g}\left[T^0{}_\alpha K^\alpha + \left(\Lambda_K^{\Sigma}{}^\alpha{}_\beta + K^\mu C^\alpha{}_{\mu\beta}\right)\Sigma^{0\beta}{}_\alpha + \left(\Lambda_K + K^\mu A_\mu\right)\cdot J^0\right]. \quad (2.33)$$

Here the integral is performed over some Cauchy slice Σ with transverse timelike coordinate x^0. Using the conservation laws (2.20) and the isometry condition (2.32), we can check that $\mathcal{H}_{\mathcal{K}}$ is conserved, i.e. it does not depend on the choice of Σ.[3]

Let us consider a generic quantum field theory coupled to an equilibrium-admitting background manifold $\mathcal{M}$. Turning on a constant global temperature $T_0 = 1/\beta_0$, the classic result in quantum statistical mechanics states that in thermodynamic equilibrium, a system can be described by a grand-canonical Gibbs density matrix

$$\rho^{\mathrm{eqb}} = \exp\left(-\beta_0\mathcal{H}_{\mathcal{K}}\right). \quad (2.34)$$

We can also write down the associated thermal partition function

$$\mathcal{Z}^{\mathrm{eqb}} = \operatorname{tr}\exp\left(-\beta_0\mathcal{H}_{\mathcal{K}}\right). \quad (2.35)$$

Here the trace is taken over a complete set of basis on the Hilbert space of our theory. Once we are provided with $\mathcal{Z}^{\mathrm{eqb}}$ as a functional of the time-independent ($\mathcal{K}$-invariant) background fields $(e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu)$, we know everything about the macroscopic behaviour of our equilibrium state. The thermal expectation values and correlators of all the macroscopic thermodynamic observables (i.e. conserved currents) can be obtained by taking functional derivatives of $\mathcal{Z}^{\mathrm{eqb}}$. To wit,

$$\langle T^\mu{}_\alpha\rangle_{\beta_0} = \frac{1}{\sqrt{-g}}\frac{\delta W^{\mathrm{eqb}}}{\delta e^\alpha{}_\mu}, \qquad \langle\Sigma^{\mu\alpha}{}_\beta\rangle_{\beta_0} = \frac{1}{\sqrt{-g}}\frac{\delta W^{\mathrm{eqb}}}{\delta C^\beta{}_{\mu\alpha}}, \qquad \langle J^\mu\rangle_{\beta_0} = \frac{1}{\sqrt{-g}}\frac{\delta W^{\mathrm{eqb}}}{\delta A_\mu}, \quad (2.36)$$

where $W^{\mathrm{eqb}} = -\ln\mathcal{Z}^{\mathrm{eqb}}$.

To compute $\mathcal{Z}^{\mathrm{eqb}}$ for a given quantum field theory, it is most natural to work in the so-called imaginary-time formalism. Let us consider that the field theory we are interested in is described at equilibrium by an effective action (see section 2.1.3) given as

$$S[e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu; \varphi^i] = \int_{\mathcal{M}} \mathrm{d}^d x\sqrt{-g}\,\mathcal{L}(e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu; \varphi^i). \quad (2.37)$$

Here φ^i are some effective dynamical degrees of freedom relevant at our energy scales. We are only interested in the time-independent configurations of the dynamical fields, therefore we choose $\delta_{\mathcal{K}}\varphi^i = 0$. Together with eq. (2.32), it implies that $K^\mu\partial_\mu\mathcal{L} = 0$. If we choose a basis $(x^\mu) = (x^0, x^i)$ on $\mathcal{M}$ such that $K^\mu = \delta^\mu{}_0$, for these configurations we have $\partial_0\mathcal{L} = 0$ and consequently

$$S[e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu; \varphi^i] = \int \mathrm{d}x^0 \int_\Sigma \mathrm{d}^{d-1}x\sqrt{-g}\,\mathcal{L}(e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu; \varphi^i). \quad (2.38)$$

[3] Actually, this statement is only correct for non-anomalous field theories. For anomalous field theories, one has to take into account the bulk piece of the Hamiltonian as well.

Here again, Σ is a Cauchy slice spanned by x^i's whose choice is irrelevant owing to the time independence of the integrand. To compute the thermal partition function, we Wick-rotate the time coordinate $x^0 \to -i\tau$ and compactify τ into a circle with period β_0. The coordinates (τ, x^i) span a Euclidean manifold $\mathcal{M}_\mathrm{E}$, which is an analytic continuation of $\mathcal{M}$. This allows us to analytically continue the effective action

$$S \to iS^{\mathrm{eqb}} = \int_{\mathcal{M}_\mathrm{E}} (-i\mathrm{d}\tau)\, \mathrm{d}^{d-1}x \left(\sqrt{-g}\,\mathcal{L}\right)_{x^0\to i\tau} = i\beta_0 \int_\Sigma \mathrm{d}^{d-1}x \sqrt{-g}\,\mathcal{L}^{\mathrm{eqb}}. \tag{2.39}$$

The superscript "eqb" in the Euclidean Lagrangian $\mathcal{L}^{\mathrm{eqb}} = -\mathcal{L}$ signifies that the background and dynamical fields making up $\mathcal{L}^{\mathrm{eqb}}$ are manifestly $\mathcal{K}$-invariant. S^{eqb} is generally known as equilibrium effective action in the literature. It can be used to compute the thermal partition function $\mathcal{Z}^{\mathrm{eqb}}$, defined as the analytic continuation of the field theory generating functional

$$\mathcal{Z} = \int \mathcal{D}\varphi^i \exp\left(iS\right) \to \mathcal{Z}^{\mathrm{eqb}} = \int \mathcal{D}\varphi^i \exp\left(-S^{\mathrm{eqb}}\right). \tag{2.40}$$

Once we have $\mathcal{Z}^{\mathrm{eqb}}$, we can go ahead and compute all the thermal expectation values and correlators. Note that the thermodynamic equilibrium is effectively described by a $(d-1)$-dimensional Euclidean quantum field theory with action S^{eqb}.

All of the discussion above was focused on finite temperature states at the thermodynamic equilibrium. That is to say that the density matrix ρ^{eqb} and the thermal partition function $\mathcal{Z}^{\mathrm{eqb}}$ are both time-independent, by the virtue of the Hamiltonian operator $\mathcal{H}_\mathcal{K}$ being foliation independent. Ideally, we would like to extend this story to arbitrary non-equilibrium states and processes, but it turns out to be quite a non-trivial task to perform. Therefore for simplicity, let us choose to work with the non-equilibrium states that are still sufficiently close to equilibrium. Let us consider a subset of states described by a set of symmetry parameters $\mathcal{B} = (\beta^\mu, \Lambda^\Sigma_\beta, \Lambda_\beta)$ that are sufficiently close to the equilibrium parameters $\mathcal{K}$. Effectively, we are looking at a field theory with dynamical fields $\varphi^I = \{\mathcal{B}, \varphi^i\}$. Similar to eq. (2.33), one can define a Hamiltonian operator $\mathcal{H}_\mathcal{B}$, however, given that $\mathcal{B}$ is not an isometry, $\mathcal{H}_\mathcal{B}$ is not conserved and depends on the choice of foliation used to define it. Consequently, the associated Gibbs density matrix $\rho = \exp(-\beta_0\mathcal{H}_\mathcal{B})$ and the partition function $\mathcal{Z} = \operatorname{tr}\exp(-\beta_0\mathcal{H}_\mathcal{B})$ are also time-dependent. We can no longer compute the time-dependent $\mathcal{Z}$ using the neat trick of imaginary-time formalism. Nevertheless, we are not really interested in the specifics of the microscopic finite-temperature field theory, but only the universal near-equilibrium behaviour. This can be quite readily captured by hydrodynamics, as we outline below.

2.2.2 Hydrodynamic fields

In the absence of an effective action describing the finite-temperature low-energy regime, we take the Noether theorem as our starting point. We require our theory of interest to respect the spacetime Poincaré transformations and some internal global G-transformations. This implies the existence of the associated conserved currents $T^\mu{}_\alpha$, $\Sigma^{\mu\alpha}{}_\beta$, and J^μ. After integrating out all the massive modes from the theory as we approach the deep IR, the only relevant degrees of freedom would be some effective massless modes φ^I along with their associated equations of motion $\mathcal{E}_I \approx 0$. For a generic low-energy theory, therefore, the conserved currents would be some expressions in terms of φ^I and background fields $(e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu)$, with appropriate transformation properties. Noether's theorem implies that these currents satisfy the identities (2.19), where the operators $\mathcal{O}^I_\alpha$, $\mathcal{O}^{I\alpha}{}_\beta$, and $\mathcal{O}^I$ encode how the fields φ^I transform under symmetries.

Due to these identities, the conservation laws (2.20) can serve as a placeholder for a vector, a $\mathfrak{so}(d,1)$-valued scalar, and a $\mathfrak{g}$-valued scalar linear combinations of the equations of motion $\mathcal{E}_I \approx 0$. That is to say that out of the dynamic fields φ^I, eq. (2.20) can serve as equations of motion for some dynamical degrees of freedom packaged into

$$\mathcal{B} = \big(\ \beta^\mu\ ,\ \ \Lambda^\Sigma_\beta\ ,\ \ \Lambda_\beta\ \big)\,. \tag{2.41}$$

We denote the remaining dynamical fields which are not captured in $\mathcal{B}$ by φ^i. The choice of this splitting of degrees of freedom near thermodynamic equilibrium is naturally motivated from our discussion in the previous subsection. In the conventional treatment of hydrodynamics, it is more convenient to rewrite $\mathcal{B}$ in terms of the so called *hydrodynamic fields*

$$\begin{gathered}\text{Velocity:}\quad u^\mu \quad \text{with} \quad u^\mu u_\mu = -1, \qquad \text{Temperature:}\quad T, \\ \text{Spin chemical potential:}\quad \mu^\Sigma \in \mathfrak{so}(d,1), \qquad G \text{ chemical potential:}\quad \mu \in \mathfrak{g},\end{gathered}$$

defined via the relations

$$\beta^\mu = \frac{1}{T}u^\mu, \qquad \Lambda^{\Sigma\alpha}_\beta{}_\beta + \beta^\mu C^\alpha{}_{\mu\beta} = \frac{1}{T}\mu^{\Sigma\alpha}{}_\beta, \qquad \Lambda_\beta + \beta^\mu A_\mu = \frac{1}{T}\mu. \tag{2.42}$$

These hydrodynamic fields, as the name suggests, form the fundamental dynamical variables of hydrodynamics. Note that a priori there is no unique definition of these fields. Therefore, like in any field theory, they are permitted to undergo arbitrary field redefinition involving themselves and the other fields φ^i and $(e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu)$ without changing any physics. Let us push this to the back our minds for the moment. We return to this issue in section 2.3.

Once we have made this logical split between the degrees of freedom, we can formally decompose $\mathcal{O}^I\mathcal{E}_I$'s appearing in eq. (2.18) into

$$\mathcal{O}^I_\alpha\mathcal{E}_I = P^{\text{ext}}_\alpha + \mathcal{O}^i_\alpha\mathcal{E}_i, \qquad \mathcal{O}^{I\alpha}{}_\beta\mathcal{E}_I = S^{\text{ext}\,\alpha}{}_\beta + \mathcal{O}^{i\alpha}{}_\beta\mathcal{E}_i, \qquad \mathcal{O}^I\mathcal{E}_I = Q^{\text{ext}} + \mathcal{O}^i\mathcal{E}_i. \tag{2.43}$$

Here P^{ext}_α, $S^{\text{ext}\,\alpha}{}_\beta$, and Q^{ext} are some linear combinations of our original equations of motion which provide dynamics for our hydrodynamic fields. Using the identities (2.19), these can also be expressed as

$$\begin{aligned}P^{\text{ext}}_\alpha &= \underline{\mathrm{D}}_\mu T^\mu{}_\alpha - e_\alpha{}^\nu\left(\mathrm{T}^\beta{}_{\nu\mu}T^\mu{}_\beta + R_{\nu\mu}{}^\gamma{}_\beta\Sigma^{\mu\beta}{}_\gamma + F_{\nu\mu}\cdot J^\mu\right) - \mathcal{O}^i_\alpha\mathcal{E}_i, \\ S^{\text{ext}\,\alpha\beta} &= \underline{\mathrm{D}}_\mu\Sigma^{\mu\alpha\beta} - T^{[\beta\alpha]} - \Sigma^{\perp\alpha\beta}_{\mathrm{H}} - \mathcal{O}^{i\alpha\beta}\mathcal{E}_i, \\ Q^{\text{ext}} &= \underline{\mathrm{D}}_\mu J^\mu - \mathrm{J}^\perp_{\mathrm{H}} - \mathcal{O}^i\mathcal{E}_i.\end{aligned} \tag{2.44}$$

From here we can interpret $P^{\rm ext}_\alpha$ as "external energy-momentum", $S^{{\rm ext}\,\alpha}{}_\beta$ as "external spin", and $Q^{\rm ext}$ as "external charge" sources, because when they are non-vanishing, possibly due to some external agents, they would violate the respective conservation equations. When the system is thermodynamically isolated, however, these sources vanish on-shell and can be taken to be the equations of motion for our hydrodynamic fields

$$P^{\rm ext}_\alpha \approx 0, \qquad S^{{\rm ext}\,\alpha}{}_\beta \approx 0, \qquad Q^{\rm ext} \approx 0. \tag{2.45}$$

Due to eq. (2.44), these equations are the same as the conservation laws (2.20) as we originally intended.

A definition: a fluid configuration is said to be *thermodynamically isolated* if the hydrodynamic fields $\mathcal{B}$ are taken on-shell by setting the external sources $P^{\rm ext}_\alpha$, $S^{{\rm ext}\,\alpha}{}_\beta$, and $Q^{\rm ext}$ to zero, while still keeping the remaining fields φ^i off-shell. Equalities in this "partially on-shell" configuration are denoted by the symbol "$\simeq$",

$$\begin{aligned}
\underline{\mathrm{D}}_\mu T^\mu{}_\alpha &\simeq e_\alpha{}^\nu \left(\mathrm{T}^\beta{}_{\nu\mu} T^\mu{}_\beta + R_{\nu\mu}{}^\gamma{}_\beta \Sigma^{\mu\beta}{}_\gamma + F_{\nu\mu}\cdot J^\mu\right) + \mathcal{O}^i_\alpha \mathcal{E}_i, \\
\underline{\mathrm{D}}_\mu \Sigma^{\mu\alpha\beta} &\simeq T^{[\beta\alpha]} + \Sigma_{\rm H}^{\perp\alpha\beta} + \mathcal{O}^{i\alpha\beta}\mathcal{E}_i, \\
\underline{\mathrm{D}}_\mu J^\mu &\simeq \mathrm{J}^\perp_{\rm H} + \mathcal{O}^i \mathcal{E}_i.
\end{aligned} \tag{2.46}$$

To this end, hydrodynamics is characterised by the most generic expressions for a set of conserved currents ($T^\mu{}_\alpha$, $\Sigma^{\mu\alpha}{}_\beta$, J^μ) and some quantities $\mathcal{E}_i$ written in terms of the hydrodynamic fields (u^μ, T, μ^Σ, μ) (or equivalently $\mathcal{B}$), other gapless fields φ^i, and background fields ($e^\alpha{}_\mu$, $C^\alpha{}_{\mu\beta}$, A_μ). These expressions are called the *hydrodynamic constitutive relations*. The fields $\mathcal{B}$ and φ^i are the dynamical field content of hydrodynamics. The equations of motion for $\mathcal{B}$ are given by eq. (2.46), while those for φ^i are given by $\mathcal{E}_i \approx 0$. This makes the system of equations closed.

2.2.3 Constitutive relations

Before we start writing down the hydrodynamic constitutive relations, we need to get some technical aspects in order. We do not know anything about the fields φ^i, and even if we did we could cook up an infinite tower of distinct tensor structures that could potentially enter the constitutive relations. Let us deal with the latter problem first. Recall that we are working under the assumption of "low-energy", which we gladly used to integrate out all the massive degrees of freedom from our theory. We must, therefore, take care that at all times we are only probing energies which are well below the energy-scale set by the first massive excitation. Heuristically, the inverse temperature defines a length scale called the mean-free path of the system. When a particle interpretation is available, the mean-free path can be understood as the average length a particle travels between collisions. We require that the perturbations in hydrodynamics are over the length scales much larger compared to this mean-free path so that the actual constituents of the theory stay irrelevant. Practically, this implies that hydrodynamics is only valid in a regime where the derivatives of background and dynamical fields are small. Within this regime, therefore, we can treat derivatives as a perturbative parameter and expand the constitutive relations order by order in derivatives. This is commonly referred to as the *derivative expansion* of hydrodynamics. At any given order in the derivative expansion, the constitutive relations only contain a finite number of tensor structures, each coming with an arbitrary multiplier function called a *transport coefficient*. Truncated to a finite derivative order, therefore, hydrodynamics is completely characterised by a finite set of transport coefficients entering the constitutive relations. At this level, these transport coefficients are completely arbitrary, and depending on the microscopic field theory at play, can be different for different hydrodynamic systems. For a given system, if the microscopic field theory is known they can be explicitly computed, or otherwise, they can be measured in an experiment. Admittedly, the number of transport coefficients grows factorially as we increase the derivative order, so hydrodynamics is only a useful framework at the first few derivative orders.

When it comes to the fields φ^i, hydrodynamics is generally evasive. It is certainly true that we need their transformation properties to make any real progress with the constitutive relations. Moreover, depending on their number and transformation properties, the spectrum of transport coefficients would be totally different. The easiest case to consider is when these gapless modes are not present at all and the hydrodynamic fields are the only relevant low-energy effective degrees of freedom. This is true of theories with a "mass-gap" where there is a finite energy difference between the ground state and the first excited state. The hydrodynamics following from here is commonly referred to as the *ordinary fluid dynamics* or simply *fluid dynamics*. Another interesting case to study is when φ^i's are Goldstone modes of a spontaneously broken symmetry, in which case we explicitly know their transformation properties. The breaking of internal symmetries leads to the curiously termed *superfluid dynamics*. On the other hand, the breaking of spacetime symmetries could lead to interesting phenomena in hydrodynamics like the formation of surfaces or momentum relaxation. Trying something more exotic, we can take φ^i to be a dynamical U(1) gauge field. This leads to the interesting physics of magnetohydrodynamics. We briefly touch upon some of the examples of these cases in the course of this work.

With some of these technicalities out of the way, we can conclude that hydrodynamics is characterised by a finite set of transport coefficients. Their actual number and contribution to the constitutive relations depend on the derivative order we are working at and what additional gapless modes we have at hand. We also commented that the explicit functional form of these transport coefficients depends on the fluid under consideration and could be obtained either by a carefully designed experiment or a field theoretic computation. If this feels rather disheartening to someone, we can, in fact, impose some empirical physical requirements on the constitutive relations and significantly bring down the number of independent transport coefficients that need to be measured. The most important of these empirical requirements, and as far as we know universally applicable, is the *local second law of thermodynamics*. We describe the statement in the section 2.3.

To avoid any confusion, we should clarify that in our entire discussion above we have only been interested in the constitutive relations and transport coefficients of a fluid. They are essentially a set of parameters characterising a fluid, such as the viscosity and conductivity, as a function of the thermodynamic variables such as the temperature and chemical potential. We also discussed how these properties could influence the velocity, temperature, or chemical potential profiles of the fluid via the conservation laws serving as equations of motion. We have not, however, discussed the actual solutions of these equations of motion, known as *fluid configurations*. There is a lot of interesting physics to be understood in these configurations, but in this work, we only focus on the fundamental principles governing fluids and their constitutive relations.

Choosing derivative order for constituent fields: To be able to use the derivative expansion when writing down the constitutive relations, we need to decide where various fields in the theory stand with respect to each other. The choice we make depends on the physical application we have in mind and affects the explicit form of the constitutive relations. But the framework of hydrodynamics works with any such choice. For reference, we always choose the temperature $T \sim \mathcal{O}(\partial^0)$. During most of this work, we are working with the choice that the fields corresponding to Abelian symmetries are $\mathcal{O}(\partial^0)$, while those corresponding to non-Abelian symmetries are $\mathcal{O}(\partial^1)$:

- Translations sector: $u^\mu, T, e^\alpha{}_\mu \sim \mathcal{O}(\partial^0)$.
- SO$(d{-}1,1)$ sector: $\mu^{\Sigma\alpha}{}_\beta, C^\alpha{}_{\mu\beta} \sim \mathcal{O}(\partial^1)$. Consequently, $\mathrm{T}^\alpha{}_{\mu\nu} \sim \mathcal{O}(\partial^1)$, $R_{\mu\nu}{}^\alpha{}_\beta \sim \mathcal{O}(\partial^2)$.
- G-sector: We split G into $G^{\mathrm{ab}} \times G^{\mathrm{nab}}$ where G^{ab} is its largest Abelian subgroup. It implies a splitting of various fields $\mu = \mu^{\mathrm{ab}} + \mu^{\mathrm{nab}}$, $A_\mu = A^{\mathrm{ab}}_\mu + A^{\mathrm{nab}}_\mu$ and $F_{\mu\nu} = F^{\mathrm{ab}}_{\mu\nu} + F^{\mathrm{nab}}_{\mu\nu}$. We choose $\mu^{\mathrm{ab}}, A^{\mathrm{ab}}_\mu \sim \mathcal{O}(\partial^0)$ and $\mu^{\mathrm{nab}}, A^{\mathrm{nab}}_\mu \sim \mathcal{O}(\partial^1)$. Consequently, $F^{\mathrm{ab}}_{\mu\nu} \sim \mathcal{O}(\partial^1)$ and $F^{\mathrm{nab}}_{\mu\nu} \sim \mathcal{O}(\partial^2)$.
- The choice for φ^i depends on their symmetry properties.

For the connections $C^\alpha{}_{\mu\beta}$ and A_μ, this choice ensures that the action of the covariant derivative operator D_μ on the constituent fields always raises a derivative order and can be treated perturbatively. We have chosen the derivative orders of the chemical potentials to match up with their connection counterparts.

We must stress that although this choice is suitable for the systems studied in this work, the hydrodynamic setup works for any arbitrary choice. For example, when focusing on hydrodynamics with spin, we typically take the spin chemical potential $\mu^\Sigma \sim \mathcal{O}(\partial^0)$; see e.g. [90]. In particular, the generalities of hydrodynamics discussed in this chapter go through irrespective of the choice of derivative orders for the constituent fields.

2.2.4 Hydrostatic principle

We started out this section with an introduction to finite temperature field theories at thermodynamic equilibrium in section 2.2.1. We motivated that in addition to the gapless dynamical modes φ^i relevant at equilibrium, near equilibrium states are described by the so-called hydrodynamic modes $\mathcal{B}$. In the limit that we approach the thermodynamic equilibrium, the fields $\mathcal{B}$ approach a set of background isometry data $\mathcal{K}$. Later in sections 2.2.2 and 2.2.3 we developed a generic formalism of hydrodynamics with dynamical fields $\mathcal{B}$ and φ^i based on symmetries and the low-energy approximation. In this subsection, we would like to subject our construction to some internal consistency conditions by requiring that the theory of hydrodynamics admits a consistent thermodynamic equilibrium.

Let us say that we have coupled our theory of hydrodynamics to an equilibrium-admitting background manifold $\mathcal{M}$ with timelike isometry $\mathcal{K}$. We can define the so-called *equilibrium fluid configurations* as solutions of the equations of motion which respect the timelike isometry, i.e. $\delta_{\mathcal{K}}\mathcal{B} = \delta_{\mathcal{K}}\varphi^i = 0$. We require that there must always exist an equilibrium fluid configuration, called the *hydrostatic configuration*, for which $\mathcal{B} = \mathcal{K}$.[4] We dub this as the *weak hydrostatic principle*. It is a very non-trivial statement, as for $\mathcal{B} = \mathcal{K}$ the hydrodynamic equations of motion (2.46) must be rendered trivial. One can imagine that this would not hold true for arbitrary hydrodynamic constitutive relations. Therefore, the weak hydrostatic principle imposes some non-trivial constraints on the form of the constitutive relations and the independent transport coefficients that can appear in it.

The weak hydrostatic principle stated above is a necessary condition for the existence of equilibrium but is still not sufficient. To guarantee the existence of a well-defined equilibrium configuration, we must require that the conserved currents and φ^i-profiles in a hydrostatic configuration can be generated from a *hydrostatic effective action* [49, 50].[5] We call this the *strong hydrostatic principle*. The effective action $S^{\text{hs}}[e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu; \varphi^i]$ is given as a functional of the fields $(e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu)$ and φ^i respecting the timelike isometry $\mathcal{K}$, defined over a spatial hypersurface. S^{hs} is a special case of a generic equilibrium effective action S^{eqb} discussed in section 2.2.1, wherein the constituent fields vary over length-scales much larger than the mean free path of the system, and hence can be written down order-by-order in derivatives. The φ^i-profiles are generated by extremising the effective action with respect to the variations in φ^i

$$\mathcal{E}_i = \frac{1}{\sqrt{-g}}\frac{\delta S^{\text{hs}}}{\delta\varphi^i} \approx 0. \tag{2.47}$$

On the other hand, the hydrostatic conserved currents are obtained by varying the action with respect to the background fields

$$T^\mu{}_\alpha\big|_{\mathcal{B}=\mathcal{K}} = \frac{1}{\sqrt{-g}}\frac{\delta S^{\text{hs}}}{\delta e^\alpha{}_\mu}, \qquad \Sigma^{\mu\alpha}{}_\beta\big|_{\mathcal{B}=\mathcal{K}} = \frac{1}{\sqrt{-g}}\frac{\delta S^{\text{hs}}}{\delta C^\beta{}_{\mu\alpha}}, \qquad J^\mu\big|_{\mathcal{B}=\mathcal{K}} = \frac{1}{\sqrt{-g}}\frac{\delta S^{\text{hs}}}{\delta A_\mu}. \tag{2.48}$$

We require the hydrostatic effective action to be invariant under $\mathcal{K}$ preserving diff $\times$ $\mathfrak{so}(d{-}1,1) \times \mathfrak{g}$ transformations, up to anomalies. The hydrodynamic equations of motion (2.46) evaluated at $\mathcal{B} = \mathcal{K}$ turn out to be the Bianchi identities associated with these symmetries for arbitrary φ^i-offshell configurations. Therefore the weak hydrostatic principle

[4]Technically, one only needs $\mathcal{B} = \mathcal{K} + \mathcal{O}(\partial)$ to define a hydrostatic configuration. But we can always fix part of the redefinition freedom in $\mathcal{B}$ to exactly set $\mathcal{B} = \mathcal{K}$ in a hydrostatic configuration. We call the class of hydrodynamic frames that satisfy this condition to be *hydrostatic frames*.

[5]When there are no other dynamical fields φ^i in the hydrodynamic description, the hydrostatic effective action reduces to a hydrostatic partition function(al). They have also been called equilibrium effective action and equilibrium partition function in the literature respectively.

follows as a corollary of the strong hydrostatic principle.

In practice, we can express S^{hs} as an integral of some scalar density $\mathcal{L}^{\text{hs}}$ made out of the $\mathcal{K}$ respecting fields $(e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu)$ and φ^i, truncated at our desired derivative order. $\mathcal{L}^{\text{hs}}$ can also contain Chern-Simons terms made out of the connections $C^\alpha{}_{\mu\beta}$ and A_μ, which are covariant under symmetries up to some boundary terms. It is the terms like these that are responsible for anomalies. Having done that, we can use eq. (2.48) to compare the hydrostatic constitutive relations with those generated using S^{hs} and read out the constraints imposed by the hydrostatic effective action. This procedure has been successfully implemented for a large number of cases in the literature [2, 49–51, 91]. In this work, however, we do not pursue this direction any further. As it turns out, all of these constraints follow from an even more restrictive physical requirement: the second law of thermodynamics, which we describe in the next section. See section 2.3.4 for the reasoning how the second law captures the strong hydrostatic principle.

2.3 Local second law of thermodynamics

2.3.1 Second law and classification of transport

Given a set of hydrodynamic constitutive relations $(T^\mu{}_\alpha, \Sigma^{\mu\alpha}{}_\beta, J^\mu, \mathcal{E}_i)$, the *local second law of thermodynamics* requires that there must exist an entropy current J^μ_{S} whose divergence is non-negative for all thermodynamically isolated fluid configurations, i.e.

$$\underline{\text{D}}_\mu J^\mu_{\text{S}} \simeq \Delta \geq 0. \tag{2.49}$$

To respect this inequality, we require that Δ evaluates to a quadratic form with positive semi-definite eigenvalues. The sacred "global" second law of thermodynamics follows from here if we define the total entropy as $S_{\text{tot}} = \int \text{d}^d x \sqrt{-g}\, J^0_{\text{S}}$. Taking a ∂_0 derivative and dropping the boundary terms,

$$\partial_0 S_{\text{tot}} = \int \text{d}^d x\, \partial_0 \left(\sqrt{-g}\, J^0_{\text{S}}\right) = \int \text{d}^d x \sqrt{-g}\, \underline{\text{D}}_\mu J^\mu_{\text{S}} \gtrsim 0, \tag{2.50}$$

we can see that the total entropy always increases. To understand why we should expect a stronger "local" version of the second law to hold, note that in hydrodynamics we are only ever leaving the equilibrium perturbatively (in derivatives). This implies that locally the system always stays in thermodynamic equilibrium while the variations away from it happen at much larger scales. So locally at every point in the spacetime, the entropy must be produced, leading to the local second law of thermodynamics. At its face value, the second law (2.49) is an inequality, but it is an inequality that must be satisfied for all thermodynamically isolated configurations. Consequently, if even a rogue implausible fluid configuration could potentially lead a transport coefficient to violate the inequality, the second law forces the said transport coefficient to zero. In this sense, the local second law of thermodynamics is a very strong statement and can significantly bring down the number of independent transport coefficients.

Note that in its original form, the second law is only defined over thermodynamically isolated fluid configurations. We can bypass this limitation by noting that eq. (2.49) can be expressed as an off-shell statement by adding to it a linear combination of the external sources

$$\underline{\mathrm{D}}_\mu J^\mu_{\mathrm{S}} + \beta^\alpha P^{\mathrm{ext}}_\alpha + \left(\Lambda^{\Sigma}_\beta{}^\beta{}_\alpha + \beta^\mu C^\beta{}_{\mu\alpha}\right) S^{\mathrm{ext}\,\alpha}{}_\beta + (\Lambda_\beta + \beta^\mu A_\mu)\cdot Q^{\mathrm{ext}} = \Delta \geq 0. \tag{2.51}$$

An important comment is in order. Recall that since the hydrodynamic fields $\mathcal{B}$ were arbitrarily defined to be solved for using the conservation laws, they admit an arbitrary redefinition freedom. In writing eq. (2.51), we have fixed this freedom off-shell by choosing the hydrodynamic fields to be equal to the multipliers for the external sources, which a priori could have been arbitrary. Note however that on-shell, this does not completely fix the redefinition freedom. We can redefine the hydrodynamic fields with arbitrary combinations of the external sources and on-shell they are still equivalent. Thanks to the hydrostatic principle, we know that the external sources identically vanish upon setting $\mathcal{B} = \mathcal{K}$ and hence can be represented entirely in terms of $\delta_{\mathcal{B}} e^\alpha{}_\mu$, $\delta_{\mathcal{B}} C^\beta{}_{\mu\alpha}$, $\delta_{\mathcal{B}} A_\mu$, $\delta_{\mathcal{B}}\varphi^i$, and their derivatives. To fix the residual redefinition freedom, therefore, all we need to do is choose a set of vector, $\mathfrak{so}(d{-}1,1)$-valued scalar, and $\mathfrak{g}$-valued scalar combinations of these constituent fields to eliminate from the constitutive relations using the equations of motion. For definiteness, in this work we choose our constitutive relations to be independent of $u^\mu \delta_{\mathcal{B}} e^\alpha{}_\mu$, $u^\mu \delta_{\mathcal{B}} C^\alpha{}_{\mu\beta}$, and $u^\mu \delta_{\mathcal{B}} A_\mu$.

As it turns out, eq. (2.51) can be expressed in a more useful form if instead of the entropy current, one works with its Legendre transform *free energy current* and the associated *free energy Hall current*

$$\begin{aligned}
N^\mu &= J^\mu_{\mathrm{S}} + \beta^\alpha T^\mu{}_\alpha + \left(\Lambda^{\Sigma}_\beta{}^\beta{}_\alpha + \beta^\mu C^\beta{}_{\mu\alpha}\right)\Sigma^{\mu\alpha}{}_\beta + (\Lambda_\beta + \beta^\mu A_\mu)\cdot J^\mu + \mathcal{N}^{i\mu}_\beta \mathcal{E}_i, \\
\mathrm{N}^\perp_{\mathrm{H}} &= \left(\Lambda^{\Sigma}_\beta{}^\beta{}_\alpha + \beta^\mu C^\beta{}_{\mu\alpha}\right)\Sigma^{\perp\alpha}_{\mathrm{H}}{}_\beta + (\Lambda_\beta + \beta^\mu A_\mu)\cdot \mathrm{J}^\perp_{\mathrm{H}},
\end{aligned} \tag{2.52}$$

where $\mathcal{N}^{i\mu}_\beta$ has been defined in eq. (2.18). Using the definition of external sources from eq. (2.44) and performing some differentiation by parts, eq. (2.51) can be expressed as the so-called *adiabaticity equation*,

$$\underline{\mathrm{D}}_\mu N^\mu - \mathrm{N}^\perp_{\mathrm{H}} = T^\mu{}_\alpha \delta_{\mathcal{B}} e^\alpha{}_\mu + \Sigma^{\mu\alpha}{}_\beta \delta_{\mathcal{B}} C^\beta{}_{\mu\alpha} + J^\mu \cdot \delta_{\mathcal{B}} A_\mu + \mathcal{E}_i \delta_{\mathcal{B}} \varphi^i + \Delta, \qquad \Delta \geq 0. \tag{2.53}$$

Here we have made use of the symmetry variations defined in eqs. (2.16) and (2.18) to simplify the expressions. This equation is at the heart of entire hydrodynamics and forms the basis for most of the discussion presented in this work.

We would like to study what constraints does imposing the second law of thermodynamics imply on the hydrodynamic constitutive relations. In other words, we would like to find the most generic hydrodynamic constitutive relations $(T^\mu{}_\alpha, \Sigma^{\mu\alpha}{}_\beta, J^\mu, \mathcal{E}_i)$ allowed by the adiabaticity equation (2.53) for some choice of the free energy current N^μ and quadratic form Δ. Conventionally, one would list the most generic tensor structures that can enter the constitutive relations and free energy current up to a particular derivative order, append them with arbitrary transport coefficients, plug them into eq. (2.53), and read out the constraints arising from demanding $\Delta \geq 0$. This is the approach we took in the example we studied in chapter 1. However, this path becomes incredibly cumbersome as we go to higher derivative orders or include non-trivial φ^i fields. Arguably, it is more fruitful to directly inspect the adiabaticity equation (2.53) and classify all the solutions it can admit. This allows us to outline a generic algorithm to generate the hydrodynamic constitutive relations with arbitrary additional gapless modes and at arbitrarily high derivative orders, without having to rely on the counting of independent tensor structures.

For the clarity of notation in the following discussion, let us define

$$\Phi = \begin{pmatrix} e^\alpha{}_\mu \\ C^\alpha{}_{\mu\beta} \\ A_\mu \\ \varphi^i \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} T^\mu{}_\alpha \\ \Sigma^{\mu\alpha}{}_\beta \\ J^\mu \\ \mathcal{E}_i \end{pmatrix}. \tag{2.54}$$

$\mathcal{C}$ and Φ can be seen as elements of an extended vector space $\mathfrak{V}$. In this notation, $\mathcal{C}$ denotes the hydrodynamic constitutive relations which are written in terms of the fields $\mathcal{B}$ and Φ. This allows us to express the adiabaticity equation (2.53) in a compact form

$$\underline{\mathrm{D}}_\mu N^\mu = \mathrm{N}_{\mathrm{H}}^{\perp} + \mathcal{C} \cdot \delta_{\mathcal{B}} \Phi + \Delta, \qquad \Delta \geq 0. \tag{2.55}$$

Looking at the form of eq. (2.55), the tensor structures appearing in the hydrodynamic constitutive relations ($T^\mu{}_\alpha$, $\Sigma^{\mu\alpha}{}_\beta$, J^μ, $\mathcal{E}_i$) can be naturally split into two independent sectors and 5 independent classes

- **Hydrostatic sector:** In this sector, constitutive relations and free energy-current are constructed out of the independent tensor structures that, or any of their linear combinations, do not vanish in a hydrostatic configuration, i.e. upon setting $\mathcal{B} = \mathcal{K}$ where $\mathcal{K}$ is a background isometry. See section 2.2.4 for details on this language. We commonly use the terminology that they or any of their linear combinations cannot contain an instance of "$\delta_{\mathcal{B}}$". Hydrostatic constitutive relations are required to satisfy a non-dissipative version of the adiabaticity equation

$$\underline{\mathrm{D}}_\mu N^\mu_{\mathrm{hs}} = \mathrm{N}_{\mathrm{H}}^{\perp} + \mathcal{C}_{\mathrm{hs}} \cdot \delta_{\mathcal{B}} \Phi. \tag{2.56}$$

  The respective solutions can be classified in three classes

  1. **Class A (anomaly induced transport)**: These are the constitutive relations which are induced by anomalies in our symmetries. They are completely fixed in terms of the anomaly polynomial, which in turn is characterised by a set of constant anomaly coefficients. See section 2.3.2.
  2. **Class $\mathbf{H_V}$ (hydrostatic vector transport):** These are the hydrostatic constitutive relations which are characterised by a free energy flow transverse to the fluid velocity, excluding Class A transport. They, as well, are completely determined up to a set of constants. See section 2.3.3.

3. **Class $\mathrm{H_S}$ (hydrostatic scalar transport):** These are the hydrostatic constitutive relations which are characterised by a free energy flow which has a component along the fluid velocity. They can be obtained from a generic hydrostatic scalar $\mathcal{N}$ which is only defined up to total derivatives. See section 2.3.4.

- **Non-hydrostatic sector:** In this sector, constitutive relations and free energy current are made out of independent tensor structures that vanish in a hydrostatic configuration. In practise, they involve all the tensor structures that have at least one instance of $\delta_{\mathscr{B}}\Phi$ or their derivatives. Being faithful to our terminology, they contain at least one instance of "$\delta_{\mathscr{B}}$". Non-hydrostatic constitutive relations are required to satisfy a non-anomalous version of the adiabaticity equation

$$\underline{\mathrm{D}}_\mu N^\mu_{\mathrm{nhs}} = \mathcal{C}_{\mathrm{nhs}} \cdot \delta_{\mathscr{B}}\Phi + \Delta, \qquad \Delta \geq 0. \tag{2.57}$$

Keep note that due to our choice while fixing the residual hydrodynamic redefinition freedom, we are not allowed to use the non-hydrostatic tensor structures which depend on $u^\mu \delta_{\mathscr{B}} e^\alpha{}_\mu$, $u^\mu \delta_{\mathscr{B}} C^\beta{}_{\mu\alpha}$, or $u^\mu \delta_{\mathscr{B}} A_\mu$. The respective solutions can be classified in two classes

4. **Class $\overline{\mathrm{D}}$ (non-dissipative transport):** These non-hydrostatic constitutive relations are characterised by no entropy production. They are completely determined by $\overline{\mathfrak{D}}_n|_{n\geq 0} \in \mathfrak{V}\times\mathfrak{V}$ with $\overline{\mathfrak{D}}_n^{\mathrm{T}} = -(-)^n\overline{\mathfrak{D}}_n$. See section 2.3.5.
5. **Class D (dissipative transport):** These non-hydrostatic constitutive relations are the only ones that are responsible for the production of entropy. They are completely determined by $\mathfrak{D}_n|_{n\geq 0} \in \mathfrak{V}\times\mathfrak{V}$ with $\mathfrak{D}_n^{\mathrm{T}} = (-)^n\mathfrak{D}_n$. See section 2.3.5.

Together, the hydrostatic and non-hydrostatic sectors make up the most generic hydrodynamic constitutive relations that respect the local second law of thermodynamics.

A comment is due on the form of the adiabaticity equation in the respective sectors. Since a hydrostatic fluid configuration is a state of thermal equilibrium, we know that there should be no entropy production in the hydrostatic sector. Consequently, Δ in the hydrostatic sector is zero. In fact, by the virtue of being a quadratic form, Δ must at least be quadratic in "$\delta_{\mathscr{B}}$". Coming to the free energy current, N^μ must also be made purely out of the hydrostatic data. It can obviously not contain multiple instances of $\delta_{\mathscr{B}}$, as the RHS of eq. (2.55) only contains one. On the other hand, if N^μ contained one $\delta_{\mathscr{B}}$, there will be a term in its gradient which has $\delta_{\mathscr{B}}$ acted upon by a derivative, which cannot be matched with the RHS either. In essence, therefore, the hydrostatic sector is completely parametrised by a hydrostatic N^μ_{hs} that satisfies eq. (2.56). Since anomalies are being taken care of in the hydrostatic sector, the non-hydrostatic sector is required to satisfy the non-anomalous adiabaticity equation (2.57).

There is also a sixth class of "hydrodynamic transport"

6. **Class S (entropy transport):** This class contains solutions to the adiabaticity equation with vanishing constitutive relations but non-trivial free energy or

entropy transport. These are completely characterised by an antisymmetric tensor $X^{\mu\nu}$ and matrices $\mathfrak{S}_{mn}|_{m,n\geq 1}$ with $\mathfrak{S}^{\mathrm{T}}_{mn} = \mathfrak{S}_{nm}$. See section 2.3.6.

However, Class S solutions are not genuine hydrodynamic transport. They merely characterise the multitude of entropy currents that satisfy the second law for the same set of constitutive relations.

In the following subsections, we cover all of these classes in detail. Most of the results presented here are directly taken from [52], which discusses the classification of the hydrodynamic transport for fluids without an independent spin current, background torsion, or extra gapless modes. The extension thereof to include φ^i fields corresponding to broken non-Abelian internal symmetries was presented in our work [1]. Work in [1] also presented a new understanding of the non-hydrostatic sector which we have adopted in this work.[6] The discussion presented here is a direct generalisation of [1] to include a spin current, background torsion, and arbitrary φ^i fields.

2.3.2 Anomaly induced transport

Before delving into the full analysis of the adiabaticity equation and the constraints it imposes on the constitutive relations, let us first deal with the affect of anomalies in hydrodynamics. The anomaly induced constitutive relations, called Class A, lie within the hydrostatic sector and show up in the hydrostatic adiabaticity equation (2.56) in the form of the free energy Hall current $\mathrm{N}^{\perp}_{\mathrm{H}}$ defined in eq. (2.52). We need to find a particular solution to eq. (2.56) for a generic $\mathrm{N}^{\perp}_{\mathrm{H}}$, which we can append to the most generic solutions of the non-anomalous adiabaticity equation to obtain the most generic hydrodynamic constitutive relations. The end result of a significant amount of literature written on this problem [47, 52, 87, 92–94] is that this particular solution can be most naturally obtained using the so-called transgression machinery. In the following we present a skimmed version of this construction.

Let us begin by defining the "shadow connections"

$$\hat{\boldsymbol{A}} = \boldsymbol{A} + \mu\,\boldsymbol{u} = (A_\mu + \mu\,u_\mu)\,\mathrm{d}x^\mu, \qquad \hat{\boldsymbol{C}}^{\alpha}{}_{\beta} = \boldsymbol{C}^{\alpha}{}_{\beta} + \mu^{\Sigma\,\alpha}{}_{\beta}\,\boldsymbol{u} = \left(C^{\alpha}{}_{\mu\beta} + \mu^{\Sigma\,\alpha}{}_{\beta}\,u_\mu\right)\mathrm{d}x^\mu. \tag{2.58}$$

These are defined so as to satisfy $\Lambda_\beta + \beta^\mu \hat{A}_\mu = \Lambda^{\Sigma}_{\beta}{}^{\alpha}{}_{\beta} + \beta^\mu \hat{C}^{\alpha}{}_{\mu\beta} = 0$. Let us denote the anomaly polynomial of the theory by $\boldsymbol{\mathcal{P}}$, which is made out of $\boldsymbol{A}$ and $\boldsymbol{C}^{\alpha}{}_{\beta}$, and can be used to obtain the Hall currents using eq. (2.25). We can also compute the associated free energy Hall current defined in eq. (2.52) using the formula

$$\star_{(d+1)}\mathbf{N}_{\mathrm{H}} = \frac{1}{T}\star_{(d+1)}\left(\mu^{\Sigma}\cdot\boldsymbol{\Sigma}_{\mathrm{H}} + \mu\cdot\mathbf{J}_{\mathrm{H}}\right). \tag{2.59}$$

[6] In [52], the non-hydrostatic and entropy transport is together classified into 5 classes: B, C, D′, $\overline{\mathrm{H}}_{\mathrm{S}}$, and $\overline{\mathrm{H}}_{\mathrm{V}}$, in contrast to our 3 classes: D, $\overline{\mathrm{D}}$, and S. Class C is characterised by terms in the entropy/free energy current which are topologically conserved, and is contained within Class S. Class D′ is equal to Class D ∪ (S \ C), which contains the solutions of the adiabaticity equation corresponding to the most generic quadratic form Δ. Finally, the three non-hydrostatic non-dissipative Classes B, $\overline{\mathrm{H}}_{\mathrm{S}}$, and $\overline{\mathrm{H}}_{\mathrm{V}}$ are an alternate (and redundant) parametrisation of Class $\overline{\mathrm{D}}$.

The "shadow anomaly polynomial" $\hat{\boldsymbol{\mathcal{P}}}$ can be obtained starting from $\boldsymbol{\mathcal{P}}$ by replacing all instances of $\boldsymbol{A}$ and $\boldsymbol{C}^\alpha{}_\beta$ with $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{C}}^\alpha{}_\beta$ respectively. A similar procedure can be used to define the shadow Hall currents as well. Having done that, let us consider

$$\star\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{P}}} = \frac{\boldsymbol{u}}{\mathrm{d}\boldsymbol{u}} \wedge \star_{(d+1)} \left(\boldsymbol{\Sigma}_\mathrm{H} - \hat{\boldsymbol{\Sigma}}_\mathrm{H}\right), \qquad \star\boldsymbol{J}_{\boldsymbol{\mathcal{P}}} = \frac{\boldsymbol{u}}{\mathrm{d}\boldsymbol{u}} \wedge \star_{(d+1)} \left(\mathbf{J}_\mathrm{H} - \hat{\mathbf{J}}_\mathrm{H}\right),$$
$$\star\boldsymbol{q}_{\boldsymbol{\mathcal{P}}} = -\frac{\boldsymbol{u}}{\mathrm{d}\boldsymbol{u}\wedge\mathrm{d}\boldsymbol{u}} \wedge \left(\boldsymbol{\mathcal{P}} - \hat{\boldsymbol{\mathcal{P}}} + T\mathrm{d}\boldsymbol{u}\wedge\star_{(d+1)}\hat{\mathbf{N}}_\mathrm{H}\right). \tag{2.60}$$

In terms of these abstract definitions, the Class A constitutive relations are given as

$$(T^{\mu\alpha})_\mathrm{A} = q^\mu_{\boldsymbol{\mathcal{P}}} u^\alpha + q^\alpha_{\boldsymbol{\mathcal{P}}} u^\mu, \qquad (\Sigma^{\mu\alpha\beta})_\mathrm{A} = \Sigma^{\mu\alpha\beta}_{\boldsymbol{\mathcal{P}}}, \qquad (J^\mu)_\mathrm{A} = J^\mu_{\boldsymbol{\mathcal{P}}}. \tag{2.61a}$$

It can be explicitly checked that they satisfy the hydrostatic adiabaticity equation (2.56) with the free energy current

$$(N^\mu)_\mathrm{A} = \frac{1}{T}\left(-q^\mu_{\boldsymbol{\mathcal{P}}} + \mu^\Sigma \cdot \Sigma^\mu_{\boldsymbol{\mathcal{P}}} + \mu\cdot J^\mu_{\boldsymbol{\mathcal{P}}}\right). \tag{2.61b}$$

Since the anomaly polynomial does not make any reference to the gapless modes φ^i, it is natural that the respective equations of motion $(\mathcal{E}_i)_\mathrm{A} = 0$ in Class A. Curiously, in Class A there is no entropy transport either

$$(J^\mu_\mathrm{S})_\mathrm{A} = 0. \tag{2.61c}$$

To verify the validity of these claims, one can define an effective action describing the anomalous transport [93]

$$S_\mathrm{A} = \int_{\mathcal{B}} \boldsymbol{\mathcal{V}}_{\boldsymbol{\mathcal{P}}}, \qquad \boldsymbol{\mathcal{V}}_{\boldsymbol{\mathcal{P}}} = \frac{\boldsymbol{u}}{\mathrm{d}\boldsymbol{u}} \wedge \left(\boldsymbol{\mathcal{P}} - \hat{\boldsymbol{\mathcal{P}}}\right). \tag{2.62}$$

The integrand $\boldsymbol{\mathcal{V}}_{\boldsymbol{\mathcal{P}}}$ is called a transgression form. After some algebraic manipulations, one can show that the infinitesimal variation of S_A is given by

$$\delta S_\mathrm{A} = \int_{\mathcal{B}} \mathrm{d}^{d+1}x\sqrt{-g_{d+1}}\left(\Sigma_\mathrm{H}{}^{\hat\mu\hat\alpha}{}_{\hat\beta}\,\delta C^{\hat\beta}{}_{\hat\mu\hat\alpha} - \hat\Sigma_\mathrm{H}{}^{\hat\mu\hat\alpha}{}_{\hat\beta}\,\delta\hat C^{\hat\beta}{}_{\hat\mu\hat\alpha} + \mathrm{J}^{\hat\mu}_\mathrm{H}\,\delta A_{\hat\mu} - \hat{\mathrm{J}}^{\hat\mu}_\mathrm{H}\,\delta\hat A_{\hat\mu}\right)$$
$$+ \int_{\mathcal{M}} \mathrm{d}^d x\sqrt{-g}\left(\Sigma^{\mu\alpha}_{\boldsymbol{\mathcal{P}}}{}_\beta\,\delta C^\beta{}_{\mu\alpha} + J^\mu_{\boldsymbol{\mathcal{P}}}\,\delta A_\mu + q^\mu_{\boldsymbol{\mathcal{P}}}\,\delta u_\mu\right). \tag{2.63}$$

The effective action S_A is manifestly invariant under all the symmetries of the theory. Consequently, under the action of an infinitesimal set of symmetry parameters $\mathfrak{X}$, the variation $\delta_\mathfrak{X} S_\mathrm{A}$ is trivially zero. In particular, if we choose $\mathfrak{X} = \alpha\mathfrak{B}$, requiring $\delta_\mathfrak{X} S_\mathrm{A} = 0$ for an arbitrary scalar field α precisely leads to the identity

$$\underline{\mathrm{D}}_\mu (N^\mu)_\mathrm{A} - \mathrm{N}^\perp_\mathrm{H} = (T^\mu{}_\alpha)_\mathrm{A}\delta_\mathfrak{B} e^\alpha{}_\mu + (\Sigma^{\mu\alpha}{}_\beta)_\mathrm{A}\delta_\mathfrak{B} C^\beta{}_{\mu\alpha} + (J^\mu)_\mathrm{A}\cdot\delta_\mathfrak{B} A_\mu, \tag{2.64}$$

which is nothing but the hydrostatic adiabaticity equation for Class A. Hence we have verified that Class A constitutive relations do satisfy the adiabaticity equation.

2.3.3 Transcendental anomalies

While we are on the topic of anomalies, let us consider a particular extension of Class A to accommodate more generic solutions of the hydrostatic adiabaticity equation. These are called *transcendental anomalies* or *hydrostatic vectors* in the literature, denoted by Class $\mathrm{H_V}$. We justify this nomenclature in the next subsection. The most natural route to these "anomalies" is using an auxiliary $\mathrm{U}(1)_\mathsf{T}$ symmetry introduced in [52]. We briefly review this method below.

Let us introduce an auxiliary Abelian global symmetry $\mathrm{U}(1)_\mathsf{T}$ in our theory and an associated background gauge field A^T_μ. We take the corresponding chemical potential to be $\mu^\mathsf{T} = T$, i.e. $\Lambda^\mathsf{T}_\beta + \beta^\mu A^\mathsf{T}_\mu = 1$. The anomaly polynomial for this enlarged theory is denoted by $\boldsymbol{\mathcal{P}}_\mathsf{T}$, which is known as the *thermal anomaly polynomial*. Generically, it is characterised by more constants compared to the physical anomaly polynomial $\boldsymbol{\mathcal{P}}$ due to the presence of additional terms involving the Chern classes of $F^\mathsf{T}_{\mu\nu}$. In the limit that $F^\mathsf{T}_{\mu\nu}$ is taken to zero, $\boldsymbol{\mathcal{P}}_\mathsf{T}$ reduces to $\boldsymbol{\mathcal{P}}$. With this in mind, let us decompose

$$\boldsymbol{\mathcal{P}}_\mathsf{T} = \boldsymbol{\mathcal{P}} + \boldsymbol{\mathcal{P}}_{\mathrm{H_V}} = \boldsymbol{\mathcal{P}} + \sum_{j\geq 1} (\boldsymbol{F}^\mathsf{T})^{\wedge j} \wedge \boldsymbol{\mathcal{P}}_{\mathrm{H_V},j}. \tag{2.65}$$

$\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}$ contains terms which are at least linear in $F^\mathsf{T}_{\mu\nu}$, wherein $\boldsymbol{\mathcal{P}}_{\mathrm{H_V},j}$ are $(d+2-2j)$-rank anomaly polynomials made out of $\boldsymbol{F}$ and $\boldsymbol{R}$. For $d = 4k-2$ and $d = 4k$, the extra piece $\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}$ involves $k(k+1)$ and $(k+1)^2$ constants respectively.

We can use the method prescribed in the previous subsection to find a particular anomaly induced solution to the extended hydrostatic adiabaticity equation

$$\underline{\mathrm{D}}_\mu N^\mu - \mathrm{N}^\perp_\mathrm{H} - \mathrm{J}^\perp_{\mathsf{T}\mathrm{H}} = T^\mu{}_\alpha \delta_\mathscr{B} e^\alpha{}_\mu + \Sigma^{\mu\alpha}{}_\beta\, \delta_\mathscr{B} C^\beta{}_{\mu\alpha} + J^\mu \cdot \delta_\mathscr{B} A_\mu + \mathcal{E}_i \delta_\mathscr{B}\varphi^i + J^\mu_\mathsf{T}\, \delta_\mathscr{B} A^\mathsf{T}_\mu, \tag{2.66}$$

where $\delta_\mathscr{B} A^\mathsf{T}_\mu = \pounds_\beta A^\mathsf{T}_\mu + \partial_\mu \Lambda^\mathsf{T}_\beta = \beta^\nu F^\mathsf{T}_{\nu\mu}$. The free energy Hall current $\mathbf{N}_\mathrm{H}$ in this extended theory is again taken to be eq. (2.59), except that $\boldsymbol{\Sigma}_\mathrm{H}$ and $\mathbf{J}_\mathrm{H}$ are now being defined using $\boldsymbol{\mathcal{P}}_\mathsf{T}$ as opposed to $\boldsymbol{\mathcal{P}}$. Similarly, $\mathbf{J}_{\mathsf{T}\mathrm{H}}$ is the Hall current associated with the $\mathrm{U}(1)_\mathsf{T}$ symmetry, i.e.

$$\star_{(d+1)} \mathbf{J}_{\mathsf{T}\mathrm{H}} = \frac{\partial \boldsymbol{\mathcal{P}}_\mathsf{T}}{\partial \boldsymbol{F}^\mathsf{T}} = \sum_{j\geq 1} j (\boldsymbol{F}^\mathsf{T})^{\wedge(j-1)} \wedge \boldsymbol{\mathcal{P}}_{\mathrm{H_V},j}. \tag{2.67}$$

Consider taking a limit of this extended theory wherein we set $F^\mathsf{T}_{\mu\nu} \to 0$. Doing this, the $\delta_\mathscr{B} A^\mathsf{T}_\mu$ term in eq. (2.66) identically vanishes. If we ignore the terms in $\boldsymbol{\mathcal{P}}_\mathsf{T}$ which are linear in $F^\mathsf{T}_{\mu\nu}$ for now (i.e. stick to $j > 1$), the $\mathrm{U}(1)_\mathsf{T}$ Hall current $\mathbf{J}_{\mathsf{T}\mathrm{H}}$ alse vanishes in this limit. Consequently, the modified adiabaticity equation (2.66) precisely reduces to the original hydrostatic adiabaticity equation (2.56). Unfortunately, for the terms in $\boldsymbol{\mathcal{P}}_\mathsf{T}$ which are linear in $F^\mathsf{T}_{\mu\nu}$ (i.e. $j = 1$), the associated Hall current $\star_{(d+1)} \mathbf{J}_{\mathsf{T}\mathrm{H}} = \boldsymbol{\mathcal{P}}_{\mathrm{H_V},1} = \mathrm{d}\boldsymbol{I}_{\mathrm{H_V},1}$ does not vanish. We can cheat our way out of this situation, however, by noting that we can shift $N^\mu \to N^\mu + (\star \boldsymbol{I}_{\mathrm{H_V},1})^\mu$ to get rid of the extra piece in the adiabaticity equation. This leads to a gauge-non-invariant free energy current, which in principle should not be an issue because the free energy or entropy currents are not physical observables of our theory and are, therefore, not restricted to be gauge invariant. However, if one insists on having a gauge-invariant entropy current, $\boldsymbol{\mathcal{P}}_{\mathrm{H_V},1}$ must be set to zero.

Modulo this technicality, we can conclude that if we are given a set of anomaly induced constitutive relations in the extended theory which satisfy eq. (2.66), we can take $F^{\mathsf{T}}_{\mu\nu} \to 0$ to gain a solution to our original adiabaticity equation (2.53). Interestingly, we find that the solution thus obtained is not merely Class A. Even though $\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}$ vanishes in this limit, the hydrodynamic constitutive relations it induces, called Class $\mathrm{H_V}$, do survive. They satisfy the non-anomalous version of the hydrostatic adiabaticity equation (2.56).

To obtain the explicit solution, let us define a shadow $\mathrm{U}(1)_{\mathsf{T}}$ gauge field $\hat{\boldsymbol{A}}^{\mathsf{T}} = \boldsymbol{A}^{\mathsf{T}} + T\boldsymbol{u}$. We can follow through the exact same analysis we did in the previous subsection, but this time with the extra Abelian gauge field, and obtain the extended anomaly induced constitutive relations. We skip the details and directly write down the solutions in $F^{\mathsf{T}}_{\mu\nu} \to 0$ limit. Since the entire construction is linear in the anomaly polynomial, we can treat $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}$ segments of the thermal anomaly polynomial $\boldsymbol{\mathcal{P}}_{\mathsf{T}}$ independently. The physical anomaly polynomial $\boldsymbol{\mathcal{P}}$ has no dependence on $F^{\mathsf{T}}_{\mu\nu}$ and leads to the familiar Class A constitutive relations we discussed in the previous subsection. The $\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}$ piece, however, is novel. The key point to note is that although $\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}$ and the associated Hall currents

$$
\begin{gathered}
\star_{(d+1)}(\boldsymbol{\Sigma}_{\mathrm{H}})_{\mathrm{H_V}} = \frac{\partial \boldsymbol{\mathcal{P}}_{\mathrm{H_V}}}{\partial \boldsymbol{R}}, \qquad \star_{(d+1)}(\mathbf{J}_{\mathrm{H}})_{\mathrm{H_V}} = \frac{\partial \boldsymbol{\mathcal{P}}_{\mathrm{H_V}}}{\partial \boldsymbol{F}}, \qquad \star_{(d+1)}(\mathbf{J}_{\mathsf{T}\mathrm{H}})_{\mathrm{H_V}} = \frac{\partial \boldsymbol{\mathcal{P}}_{\mathrm{H_V}}}{\partial \boldsymbol{F}^{\mathsf{T}}}, \\
\star_{(d+1)}(\mathbf{N}_{\mathrm{H}})_{\mathrm{H_V}} = \frac{1}{T}\star_{(d+1)}\left(\mu^{\Sigma}\cdot(\boldsymbol{\Sigma}_{\mathrm{H}})_{\mathrm{H_V}} + \mu\cdot(\mathbf{J}_{\mathrm{H}})_{\mathrm{H_V}}\right),
\end{gathered} \tag{2.68}
$$

vanish upon setting $F^{\mathsf{T}}_{\mu\nu} \to 0$ (except $j=1$), the respective shadow quantities do not. These are precisely responsible for the non-trivial Class $\mathrm{H_V}$ constitutive relations. We find

$$
\begin{gathered}
(T^{\mu\alpha})_{\mathrm{H_V}} = q^{\mu}_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} u^{\alpha} + q^{\alpha}_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} u^{\mu}, \qquad (\Sigma^{\mu\alpha\beta})_{\mathrm{H_V}} = \Sigma^{\mu\alpha\beta}_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}}, \qquad (J^{\mu})_{\mathrm{H_V}} = J^{\mu}_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}}, \\
(N^{\mu})_{\mathrm{H_V}} = J^{\mu}_{\mathsf{T}\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} + \frac{1}{T}\left(-q^{\mu}_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} + \mu^{\Sigma}\cdot\Sigma_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} + \mu\cdot J^{\mu}_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}}\right) - (\star\boldsymbol{I}_{\mathrm{H_V},1})^{\mu},
\end{gathered} \tag{2.69}
$$

along with $(\mathcal{E}_i)_{\mathrm{H_V}} = 0$, where

$$
\begin{gathered}
\star\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} = -\frac{\boldsymbol{u}}{\mathrm{d}\boldsymbol{u}}\wedge\star_{(d+1)}(\hat{\boldsymbol{\Sigma}}_{\mathrm{H}})_{\mathrm{H_V}}\Big|_{\boldsymbol{F}^{\mathsf{T}}\to 0}, \qquad \star\boldsymbol{J}_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} = -\frac{\boldsymbol{u}}{\mathrm{d}\boldsymbol{u}}\wedge\star_{(d+1)}(\hat{\mathbf{J}}_{\mathrm{H}})_{\mathrm{H_V}}\Big|_{\boldsymbol{F}^{\mathsf{T}}\to 0}, \\
\star\boldsymbol{J}_{\mathsf{T}\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} = \frac{\boldsymbol{u}}{\mathrm{d}\boldsymbol{u}}\wedge\star_{(d+1)}\left((\mathbf{J}_{\mathsf{T}\mathrm{H}})_{\mathrm{H_V}} - (\hat{\mathbf{J}}_{\mathsf{T}\mathrm{H}})_{\mathrm{H_V}}\right)\Big|_{\boldsymbol{F}^{\mathsf{T}}\to 0}, \\
\star\boldsymbol{q}_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} = \frac{\boldsymbol{u}}{\mathrm{d}\boldsymbol{u}\wedge\mathrm{d}\boldsymbol{u}}\wedge\left(\hat{\boldsymbol{\mathcal{P}}}_{\mathrm{H_V}} - T\mathrm{d}\boldsymbol{u}\wedge\star_{(d+1)}\left((\hat{\mathbf{N}}_{\mathrm{H}})_{\mathrm{H_V}} + (\hat{\mathbf{J}}_{\mathsf{T}\mathrm{H}})_{\mathrm{H_V}}\right)\right)\Big|_{\boldsymbol{F}^{\mathsf{T}}\to 0}.
\end{gathered} \tag{2.70}
$$

The Class $\mathrm{H_V}$ constitutive relations satisfy the non-anomalous version of the adiabaticity equation (2.56). Unlike Class A however, these constitutive relations do induce a non-trivial entropy transport

$$
(J^{\mu}_{\mathrm{S}})_{\mathrm{H_V}} = J^{\mu}_{\mathsf{T}\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}}. \tag{2.71}
$$

Finally, following our discussion around eq. (2.62), we can also write down an effective action to generate Class $\mathrm{H_V}$ constitutive relations

$$
S_{\mathrm{H_V}} = -\int_{\mathcal{B}} \frac{\boldsymbol{u}}{\mathrm{d}\boldsymbol{u}}\wedge\hat{\boldsymbol{\mathcal{P}}}_{\mathrm{H_V}}\Big|_{\boldsymbol{F}^{\mathsf{T}}\to 0}. \tag{2.72}
$$

Varying this action and invoking its invariance under symmetries, we can verify that the Class $\mathrm{H_V}$ constitutive relations indeed satisfy the non-anomalous adiabaticity equation.

2.3.4 Hydrostatic transport

Over the previous two subsections, we have covered two special classes of solutions of the hydrostatic adiabaticity equation. First, the Class A constitutive relations form a particular solution to eq. (2.56) which takes care of all the anomaly induced hydrodynamic transport. Second, the Class $\mathrm{H_V}$ constitutive relations satisfy the non-anomalous version of eq. (2.56) and are characterised by a set of undermined constants. We are now in a position to specify the remaining hydrostatic constitutive relations as well.

We would first like to note that there can be no non-trivial solutions to the hydrostatic adiabaticity equation (2.56) with $N^\mu_{\text{hs}} = 0$. Consequently, the hydrostatic constitutive relations are characterised by the most generic hydrostatic free energy current N^μ_{hs}. We choose a decomposition

$$N^\mu_{\text{hs}} = \left(\mathcal{N}\beta^\mu + \Theta^\mu_{\mathcal{N}}\right) + \mathbb{N}^\mu, \tag{2.73}$$

where $\mathbb{N}^\mu\beta_\mu = 0$. $\mathcal{N}$ is the most generic scalar made out of the independent hydrostatic data. $\Theta^\mu_{\mathcal{N}}$ is a $\mathcal{N}$-dependent non-hydrostatic vector defined via

$$\underline{\mathrm{D}}_\mu(\mathcal{N}\beta^\mu) = \frac{1}{\sqrt{-g}}\delta_{\mathscr{B}}\left(\sqrt{-g}\,\mathcal{N}\right) = \Phi\cdot\mathcal{C}_{\mathrm{H_S}} - \underline{\mathrm{D}}_\mu\Theta^\mu_{\mathcal{N}}, \tag{2.74}$$

which ensures that $\underline{\mathrm{D}}_\mu(\mathcal{N}\beta^\mu + \Theta^\mu_{\mathcal{N}})$ has a bare "$\delta_{\mathscr{B}}$" to match the RHS of eq. (2.53). Eq. (2.74) also defines the constitutive relations $\mathcal{C}_{\mathrm{H_S}}$ associated with $\mathcal{N}$, called the Class $\mathrm{H_S}$ constitutive relations. The respective free energy current is simply $(N^\mu)_{\mathrm{H_S}} = \mathcal{N}\beta^\mu + \Theta^\mu_{\mathcal{N}}$.

On the other hand, $\mathbb{N}^\mu$ is the most generic hydrostatic vector transverse to β^μ, such that $\underline{\mathrm{D}}_\mu\mathbb{N}^\mu - \mathrm{N}^\perp_{\mathrm{H}}$ has exactly one bare "$\delta_{\mathscr{B}}$", so that it can fit the hydrostatic adiabaticity equation. We have already seen two classes of free energy currents which meet this criteria in sections 2.3.2 and 2.3.3,

$$\mathbb{N}^\mu = (N^\mu)_{\mathrm{A}} + (N^\mu)_{\mathrm{H_V}}. \tag{2.75}$$

The corresponding constitutive relations $\mathcal{C}_{\mathrm{A}}$ and $\mathcal{C}_{\mathrm{H_V}}$ are given in eq. (2.61) and eq. (2.69) respectively. This also justifies the name "hydrostatic vectors" for Class $\mathrm{H_V}$ in contrast to the "hydrostatic scalars" for Class $\mathrm{H_S}$. As it happens, eq. (2.75) is already the most generic form of $\mathbb{N}^\mu$ allowed by the requirements laid out. However, we refrain from giving a detailed proof here; interested readers can consult [52, 87, 92]. Therefore, Classes A, $\mathrm{H_V}$, and $\mathrm{H_S}$ make up the complete set of hydrostatic constitutive relations.

By definition, the hydrostatic constitutive relations completely determine the physics in a hydrostatic fluid configuration (obtained by setting $\mathscr{B} = \mathscr{K}$ where $\mathscr{K}$ is a timelike background isometry). But as we discussed in section 2.2.4, to maintain consistency with the thermodynamic equilibrium, constitutive relations in a hydrostatic configuration must admit a hydrostatic effective action. Luckily for us, this notion is already hardwired in the second law requirement. Indeed, the Class $\mathrm{H_S}$ constitutive relations in a hydrostatic configuration can be generated from an effective action [51]

$$S^{\text{hs}}_{\text{H}_\text{S}} = \beta_0 \int_\Sigma \mathrm{d}^{d-1}x\sqrt{-g}\,(\mathcal{N})_{\mathcal{B}=\mathcal{K}}\,. \tag{2.76}$$

The hydrostatic effective actions for Classes A and H_V, on the other hand, are trivially obtained by setting $\mathcal{B} = \mathcal{K}$ in eqs. (2.62) and (2.72) respectively. Furthermore, the hydrostatic effective action thus obtained is actually the most generic such effective action we can write down in a hydrostatic configuration. Therefore, within the hydrostatic sector, the constraints arising form requiring the hydrostatic principle or those following from the second law are exactly equivalent.

2.3.5 Non-hydrostatic transport

Having figured out the hydrostatic transport, let us now move on to the non-hydrostatic sector. To remind ourselves, these are the constitutive relations which contain at least one instance of $\delta_\mathcal{B}\Phi$ or their derivatives. To this end, let us introduce some new notation. Let us define a totally symmetric covariant derivative operator

$$\mathrm{D}^n = \left(\mathrm{D}_{(\mu_1}\mathrm{D}_{\mu_2}\ldots\mathrm{D}_{\mu_n)}\right). \tag{2.77}$$

D^n forms a basis for all the differential operators, as the antisymmetric derivatives can always be replaced by combinations involving curvatures and field strengths. Given a differential operator $\mathcal{O}$, we define its conjugate $\mathcal{O}^\dagger$ via

$$X_1\,(\mathcal{O}X_2) = \left(\mathcal{O}^\dagger X_1\right)X_2 + \underline{\mathrm{D}}_\mu\,(\cdots)^\mu\,, \tag{2.78}$$

where X_1, X_2 are some arbitrary fields. Expressed in terms of the basis operators, we have $(\mathrm{D}^n)^\dagger = (-)^n\mathrm{D}^n$.

With this new derivative operator in place, the most generic non-hydrostatic constitutive relations can be expressed in a compact form

$$\mathcal{C}_{\text{nhs}} = -\sum_{n=0}^{\infty}\frac{1}{2}\Big(\mathfrak{C}_n\cdot(\mathrm{D}^n\delta_\mathcal{B}\Phi) + \mathrm{D}^n(\mathfrak{C}_n\cdot\delta_\mathcal{B}\Phi)\Big). \tag{2.79}$$

$\mathfrak{C}_n \in \mathfrak{V}\times\mathfrak{V}$ are matrices with additional n symmetric indices to be contracted with D^n. The last term in eq. (2.79) is taken purely for convenience and can be absorbed into the first via differentiation by parts. Let us factor $\mathcal{C}_{\text{nhs}}$ into a so-called dissipative (Class D) and a non-dissipative (Class $\overline{\text{D}}$) class parametrised by

$$\mathfrak{D}_n = \frac{1}{2}\left(\mathfrak{C}_n + (-)^n\mathfrak{C}_n^{\mathrm{T}}\right), \qquad \overline{\mathfrak{D}}_n = \frac{1}{2}\left(\mathfrak{C}_n - (-)^n\mathfrak{C}_n^{\mathrm{T}}\right), \tag{2.80}$$

respectively. Beginning with the non-dissipative piece first, as it is simpler, we find that

$$\mathcal{C}_{\overline{\mathrm{D}}}\cdot\delta_\mathcal{B}\Phi = \underline{\mathrm{D}}_\mu\Theta^\mu_{\overline{\mathfrak{D}}}, \tag{2.81}$$

where $\underline{\mathrm{D}}_\mu \Theta^\mu_{\overline{\mathfrak{D}}}$ is a total derivative piece obtained after successive differentiation by parts. Comparing it to eq. (2.57) we can infer that the Class $\overline{\mathrm{D}}$ constitutive relations identically satisfy the adiabaticity equation with $(N^\mu)_{\overline{\mathrm{D}}} = \Theta^\mu_{\overline{\mathfrak{D}}}$ and $\Delta_{\overline{\mathrm{D}}} = 0$, with no constraints imposed. Hence the name non-dissipative.

Moving on to the more non-trivial dissipative constitutive relations, we instead find

$$\mathcal{C}_{\mathrm{D}} \cdot \delta_{\mathscr{B}}\Phi = -\sum_{n=0}^{\infty} \delta_{\mathscr{B}}\Phi \cdot \mathfrak{D}_n \cdot (\mathrm{D}^n \delta_{\mathscr{B}}\Phi) + \underline{\mathrm{D}}_\mu \Theta^\mu_{\mathfrak{D},0}. \tag{2.82}$$

To muster it into a form apt for the adiabaticity equation, we need to do a little more work. We start by noting that the above equation can be massaged into

$$\begin{aligned}\mathcal{C}_{\mathrm{D}} \cdot \delta_{\mathscr{B}}\Phi &= -\delta_{\mathscr{B}}\Phi \cdot \boldsymbol{\eta} \cdot \delta_{\mathscr{B}}\Phi - 2\delta_{\mathscr{B}}\Phi \cdot \boldsymbol{\eta} \cdot (\Upsilon_1 \cdot \delta_{\mathscr{B}}\Phi) + \underline{\mathrm{D}}_\mu \Theta^\mu_{\mathfrak{D},0} \\ &= -\underbrace{\Big((1+\Upsilon_1)\cdot\delta_{\mathscr{B}}\Phi\Big) \cdot \boldsymbol{\eta} \cdot \Big((1+\Upsilon_1)\cdot\delta_{\mathscr{B}}\Phi\Big)}_{\text{quadratic form}} \\ &\qquad + \underbrace{(\Upsilon_1 \cdot \delta_{\mathscr{B}}\Phi)^{\mathrm{T}} \cdot \boldsymbol{\eta} \cdot (\Upsilon_1 \cdot \delta_{\mathscr{B}}\Phi)}_{\text{residue}} + \underbrace{\underline{\mathrm{D}}_\mu \Theta^\mu_{\mathfrak{D},0}}_{\text{total derivative}},\end{aligned} \tag{2.83}$$

where $\mathfrak{D}_{0(n)}$ denotes the nth derivative piece in $\mathfrak{D}_{(0)}$ and $\boldsymbol{\eta} = \mathfrak{D}_{0(0)}$. On the other hand, Υ_1 is a differential operator

$$\Upsilon_1 = \frac{1}{2}\boldsymbol{\eta}^{-1} \sum_{n=1}^{\infty} \left(\mathfrak{D}_{0(n)} + \mathfrak{D}_n \mathrm{D}^n\right). \tag{2.84}$$

The quadratic form piece in eq. (2.83) is of most interest to us, as it contributes to Δ. The total derivative piece on the other hand is a contribution to the free energy current N^μ. However, we would like to get rid of the residue piece, which has at least 4 derivatives. Using differentiation by parts, this piece can be rewritten as

$$(\Upsilon_1 \cdot \delta_{\mathscr{B}}\Phi) \cdot \boldsymbol{\eta} \cdot (\Upsilon_1 \cdot \delta_{\mathscr{B}}\Phi) = \delta_{\mathscr{B}}\Phi \cdot \Big(\Upsilon_1^\dagger \cdot \boldsymbol{\eta} \cdot \Upsilon_1 \cdot \delta_{\mathscr{B}}\Phi\Big) + \underline{\mathrm{D}}_\mu \Theta^\mu_{\mathfrak{D},1}. \tag{2.85}$$

Putting this back in eq. (2.83) we get

$$\begin{aligned}\mathcal{C}_{\mathrm{D}} \cdot \delta_{\mathscr{B}}\Phi = &-\underbrace{\Big((1+\Upsilon_1+\Upsilon_2)\cdot\delta_{\mathscr{B}}\Phi\Big) \cdot \boldsymbol{\eta} \cdot \Big((1+\Upsilon_1+\Upsilon_2)\cdot\delta_{\mathscr{B}}\Phi\Big)}_{\text{quadratic form}} \\ &+ \underbrace{\Big((2\Upsilon_1+\Upsilon_2)\cdot\delta_{\mathscr{B}}\Phi\Big) \cdot \boldsymbol{\eta} \cdot (\Upsilon_2 \cdot \delta_{\mathscr{B}}\Phi)}_{\text{residue}} + \underbrace{\underline{\mathrm{D}}_\mu \Big(\Theta^\mu_{\mathfrak{D},0} + \Theta^\mu_{\mathfrak{D},1}\Big)}_{\text{total derivative}},\end{aligned} \tag{2.86}$$

where Υ_2 is another differential operator

$$\Upsilon_2 = -\frac{1}{2}\boldsymbol{\eta}^{-1} \cdot \Upsilon_1^\dagger \cdot \boldsymbol{\eta} \cdot \Upsilon_1. \tag{2.87}$$

Comparing eq. (2.86) to eq. (2.83), hopefully the reader can make out a repeating pattern. The quadratic form piece now has some additional higher derivative terms, whereas we have pushed the residue piece to 5th derivative order. We can repeat this procedure iteratively to push the residue piece to arbitrarily high derivative orders and obtain

$$\mathcal{C}_{\mathrm{D}} \cdot \delta_{\mathscr{B}}\Phi = -\left(\Upsilon \cdot \delta_{\mathscr{B}}\Phi\right) \cdot \boldsymbol{\eta} \cdot \left(\Upsilon \cdot \delta_{\mathscr{B}}\Phi\right) + \underline{\mathrm{D}}_\mu \Theta^\mu_{\mathfrak{D}}, \tag{2.88}$$

where

$$\Upsilon_{i+1}\big|_{i=1}^{\infty} = -\boldsymbol{\eta}^{-1} \cdot \left(\sum_{k=1}^{d-1} \Upsilon_k^\dagger + \frac{1}{2}\Upsilon_i^\dagger\right)\left(\boldsymbol{\eta}\cdot\Upsilon_i\right), \qquad \Upsilon = 1 + \sum_{n=1}^{\infty}\Upsilon_n, \tag{2.89}$$

and $\Theta^M_{\mathfrak{D}} = \sum_{n=0}^{\infty}\Theta^M_{\mathfrak{D},n}$. Finally, comparing eq. (2.88) to eq. (2.57), we can see that Class D constitutive relations satisfy the adiabaticity equation with

$$(N^\mu)_{\mathrm{D}} = \Theta^\mu_{\mathfrak{D}}, \qquad \Delta_{\mathrm{D}} = \left(\Upsilon \cdot \delta_{\mathscr{B}}\Phi\right) \cdot \boldsymbol{\eta} \cdot \left(\Upsilon \cdot \delta_{\mathscr{B}}\Phi\right). \tag{2.90}$$

The second law requirement $\Delta \geq 0$ only gives a constraint on the first derivative non-hydrostatic constitutive relations by forcing all the eigenvalues of $\boldsymbol{\eta} = \mathfrak{D}_{0(0)}$ to be non-negative. Apart from these inequalities, we do not get any constraints from the second law in the non-hydrostatic sector. In particular, unlike the hydrostatic sector, the second law does not switch off any transport coefficients. This argument was first presented by Sayantani Bhattacharyya in [91, 95]. Our presentation above can be seen as a succinct summary of her seminal work.

2.3.6 Entropy transport

In the previous subsections we have taken up the task of classifying the most generic hydrodynamic constitutive relations $\mathcal{C}$, which admit a free energy current N^μ and a quadratic form Δ such that the adiabaticity equation (2.55) is satisfied. However, the N^μ and Δ which achieve this goal for a given $\mathcal{C}$ do not have to be unique. In fact, we can shift N^μ and Δ with an arbitrary solution of

$$\underline{\mathrm{D}}_\mu (N^\mu)_{\mathrm{S}} = \Delta_{\mathrm{S}}, \tag{2.91}$$

and still satisfy the adiabaticity equation for the same set of constitutive relations $\mathcal{C}$. The solutions of eq. (2.91) are called Class S "constitutive relations". They satisfy the non-anomalous adiabaticity equation (2.55) with vanishing constitutive relations $\mathcal{C}_{\mathrm{S}} = 0$. The associated entropy transport however is non-trivial

$$(J^\mu_{\mathrm{S}})_{\mathrm{S}} = (N^\mu)_{\mathrm{S}}, \tag{2.92}$$

hence the name *entropy transport*. It should be noted however that Class S constitutive relations are not genuine hydrodynamic transport, they merely parametrise the multitude of entropy/free energy currents which meet the second law requirement for the same set of constitutive relations.

Firstly, there are some trivial solutions in Class S. Consider an arbitrary antisymmetric tensor $X^{\mu\nu}$. If we take $(N^\mu)_{\mathrm{S}} = \underline{\mathrm{D}}_\nu X^{\mu\nu} + \frac{1}{2}\mathrm{T}^\mu{}_{\nu\rho}X^{\nu\rho}$, the divergence $\underline{\mathrm{D}}_\mu (N^\mu)_{\mathrm{S}}$ is identically zero and satisfies eq. (2.91) with $\Delta_{\mathrm{S}} = 0$. To obtain the remaining non-trivial solutions, let us start with the most generic allowed quadratic form

$$\Delta_{\rm S} = \sum_{m,n\geq 0} (\mathrm{D}^m \delta_{\mathscr{B}}\Phi) \cdot \mathfrak{S}_{mn} \cdot (\mathrm{D}^n \delta_{\mathscr{B}}\Phi), \tag{2.93}$$

where $\mathfrak{S}_{mn} \in \mathfrak{V} \times \mathfrak{V}$ satisfy $\mathfrak{S}^{\rm T}_{mn} = \mathfrak{S}_{nm}$. We know that an arbitrary $\mathfrak{S}_{mn}$ cannot correspond to the Class S constitutive relations, as some part of it gives rise to the Class D transport. As we saw in section 2.3.5, the Class D transport is characterised by a quadratic form given in terms of a matrix $\mathfrak{D}_n$. So to pass as valid Class S transport, we must eliminate $\mathfrak{D}_n$ worth of components from $\mathfrak{S}_{mn}$. To see this, let us perform successive differentiation by parts on eq. (2.93) and obtain

$$\begin{aligned}\Delta_{\rm S} &= \delta_{\mathscr{B}}\Phi \cdot \mathfrak{S}_{00} \cdot \delta_{\mathscr{B}}\Phi + 2\sum_{n\geq 1} \delta_{\mathscr{B}}\Phi \cdot \mathfrak{S}_{0n} \cdot (\mathrm{D}^n \delta_{\mathscr{B}}\Phi) + \sum_{m,n\geq 1} (\mathrm{D}^m \delta_{\mathscr{B}}\Phi) \cdot \mathfrak{S}_{mn} \cdot (\mathrm{D}^n \delta_{\mathscr{B}}\Phi) \\ &= \delta_{\mathscr{B}}\Phi \cdot \mathfrak{S}_{00} \cdot \delta_{\mathscr{B}}\Phi + 2\delta_{\mathscr{B}}\Phi \cdot \mathfrak{S}_{01} \cdot (\mathrm{D}^1 \delta_{\mathscr{B}}\Phi) + \sum_{n\geq 2} \delta_{\mathscr{B}}\Phi \cdot \Big(2\mathfrak{S}_{0n} + \mathfrak{S}_n\Big) \cdot (\mathrm{D}^n \delta_{\mathscr{B}}\Phi) \\ &\quad + \underline{\mathrm{D}}_\mu \Theta^\mu_{\mathfrak{S}}. \end{aligned} \tag{2.94}$$

In the second step, we have isolated a series of differential operators $\mathfrak{S}_n$ via the relation

$$\sum_{m,n\geq 1} (-)^m \mathrm{D}^m \Big(\mathfrak{S}_{mn} \cdot (\mathrm{D}^n \delta_{\mathscr{B}}\Phi)\Big) \equiv \sum_{n\geq 2} \mathfrak{S}_n \cdot (\mathrm{D}^n \delta_{\mathscr{B}}\Phi). \tag{2.95}$$

If we choose $\mathfrak{S}_{00} = \mathfrak{S}_{01} = 0$ and $\mathfrak{S}_{0n} = -\mathfrak{S}_n/2$ for $n \geq 2$, we can satisfy eq. (2.91) with $(N^\mu)_{\rm S} = \Theta^\mu_{\mathfrak{S}}$. These are the $\mathfrak{D}_m$ worth of conditions on $\mathfrak{S}_{mn}$ we were alluding to. Finally,

$$(N^\mu)_{\rm S} = \underline{\mathrm{D}}_\nu X^{\mu\nu} + \frac{1}{2}\mathrm{T}^\mu{}_{\nu\rho} X^{\nu\rho} + \Theta^\mu_{\mathfrak{S}}, \tag{2.96}$$

parametrises the most generic Class S constitutive relations.

2.4 Summary and torsionless limit

This chapter has been quite technical, so let us summarise the important points. Hydrodynamics is characterised by its constitutive relations: the most generic expressions of $(T^\mu{}_\alpha, \Sigma^{\mu\alpha}{}_\beta, J^\mu, \mathcal{E}_i)$ in terms of the dynamical fields $(u^\mu, T, \mu^\Sigma, \mu)$ (or $\mathscr{B}$) and φ^i, and the background fields $(e^\alpha{}_\mu, C^\alpha{}_{\mu\beta}, A_\mu)$, arranged in a derivative expansion. The equations of motion for the dynamical fields are given by eq. (2.46) and $\mathcal{E}_i \approx 0$. Up to a given derivative order, the constitutive relations contain all the possible tensor structures made out of its constituent fields and their derivatives, multiplied with arbitrary transport coefficients. The actual number of such transport coefficients, however, depends on the possible tensor structures we can write down, which in turn depends on the derivative order we are working at and on the number and transformation properties of the gapless modes φ^i. In general, the explicit functional form of these transport coefficients depends on the system in question, but we can bring the number down considerably by imposing the second law of thermodynamics. It requires that there must exist an entropy current $J^\mu_{\rm S}$, or equivalently a free energy current N^μ, such that the adiabaticity equation (2.53) is satisfied for some quadratic form $\Delta \geq 0$. This requirement constrains the hydrodynamic constitutive relations to be in one of the five classes

$$\mathcal{C} = \mathcal{C}_{\text{hs}} + \mathcal{C}_{\text{nhs}} = (\mathcal{C}_{\text{A}} + \mathcal{C}_{\text{H}_\text{V}} + \mathcal{C}_{\text{H}_\text{S}}) + \big(\mathcal{C}_{\overline{\text{D}}} + \mathcal{C}_{\text{D}}\big)\,. \tag{2.97}$$

In addition, there is also a Class S worth of redundancies in the entropy or free energy currents we can choose from, to satisfy the second law for a given set of constitutive relations.

For most applications of hydrodynamics, it is useful to work with the energy-momentum tensor and charge current only and no spin current. The spin current does not necessarily have to be zero. We could just choose not to probe this particular observable by turning off the background torsion. Without torsion, we can also fall back to the good old metric formulation with background fields $(g_{\mu\nu}, A_\mu)$, as opposed to the vielbein formulation that we have been working with. This is the premise in which most of the work in relativistic field theories, especially hydrodynamics, is done. In this setting, we can use the second equation in the relativistic identities (2.19) to eliminate the antisymmetric part of the energy-momentum tensor from the first equation. Once that is done, we can show that remaining equations can be rewritten as

$$\begin{aligned}
\mathrm{D}_\mu T^{\mu\nu}_{\text{B}} &= F^{\nu\rho}\cdot J_\rho + \mathrm{D}_\mu \Sigma^{\perp\nu\mu}_{\text{H}} + \mathcal{O}^I_\alpha \mathcal{E}_I, \\
\mathrm{D}_\mu J^\mu &= \mathrm{J}^\perp_{\text{H}} + \mathcal{O}^I \mathcal{E}_I,
\end{aligned} \tag{2.98}$$

where we have defined the so-called symmetric *Belinfante energy-momentum tensor*

$$T^{\mu\nu}_{\text{B}} = T^{(\mu\nu)} + 2\,\mathrm{D}_\rho \Sigma^{(\mu\nu)\rho}. \tag{2.99}$$

This is the quantity that couples to the spacetime metric and appears in the Einstein's equations in general relativity. Later in the text, when focusing on these theories without a spin current, we drop the subscript B in $T^{\mu\nu}_{\text{B}}$ for clarity. The corresponding conservation equations on the other hand are given as

$$\begin{aligned}
\mathrm{D}_\mu T^{\mu\nu}_{\text{B}} &\approx F^{\nu\rho}\cdot J_\rho + \mathrm{D}_\mu \Sigma^{\perp\nu\mu}_{\text{H}}, \\
\mathrm{D}_\mu J^\mu &\approx \mathrm{J}^\perp_{\text{H}},
\end{aligned} \tag{2.100}$$

which are perhaps more familiar to the reader. In the flat space limit, they simply read

$$\partial_\mu T^{\mu\nu}_{\text{B}} \approx 0, \qquad \partial_\mu J^\mu \approx 0. \tag{2.101}$$

An interesting thing to note is that what used to be a Lorentz anomaly $\Sigma^{\perp\alpha\beta}_{\text{H}}$ in the spin conservation, shows up as a gravitational anomaly $\mathrm{D}_\mu \Sigma^{\perp\nu\mu}_{\text{H}}$ in the Belinfante energy-momentum conservation. This is the incarnation in which this anomaly appears in most of the literature.

Moving on to hydrodynamics, the hydrodynamic fields are taken to be merely (u^μ, T, μ)

or equivalently $\mathcal{B} = (\beta^\mu, \Lambda_\beta)$. Formally, we can think that the spin chemical potential μ^Σ has been "solved for" using the spin conservation equation in terms of the other fields in the theory, and has been substituted back into $T^{\mu\nu}_{\mathrm{B}}$, J^μ, and $\mathcal{E}^i$. The associated analogue to the adiabaticity equation (2.53) is given by

$$\mathrm{D}_\mu N^\mu - \mathrm{N}^{\perp}_{\mathrm{H}} = \frac{1}{2} T^{\mu\nu}_{\mathrm{B}} \delta_{\mathcal{B}} g_{\mu\nu} + J^\mu \cdot \delta_{\mathcal{B}} A_\mu + \mathcal{E}_i \delta_{\mathcal{B}} \varphi^i + \Delta, \qquad \Delta \geq 0. \tag{2.102}$$

If we define

$$\Phi = \begin{pmatrix} \frac{1}{2} g_{\mu\nu} \\ A_\mu \\ \varphi^i \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} T^{\mu\nu} \\ J^\mu \\ \mathcal{E}_i \end{pmatrix}, \tag{2.103}$$

we can re-express the adiabaticity equation into

$$\mathrm{D}_\mu N^\mu = \mathrm{N}^{\perp}_{\mathrm{H}} + \mathcal{C} \cdot \delta_{\mathcal{B}} \Phi + \Delta, \qquad \Delta \geq 0, \tag{2.104}$$

which is essentially the same as our previous version in eq. (2.55). Therefore, all our discussion on hydrodynamic classification in section 2.3 still holds true.

This concludes our discussion of the principles of relativistic hydrodynamics. In the next chapter, we consider some explicit applications of these ideas.

3 | Applications: relativistic hydrodynamics

In chapter 2 we studied the fundamentals of relativistic hydrodynamics in an abstract language. In this chapter, we take these key ideas and apply them to some explicit examples. We start with the well-known example of ordinary relativistic fluids and make our way to the more involved relativistic superfluids. Later, we also briefly talk about relativistic fluids with surfaces. For concreteness, we stick to $d = 4$ spacetime dimensions in this chapter.

3.1 | Relativistic fluids

Ordinary relativistic fluids are the simplest hydrodynamic systems we can consider. They do not contain any additional gapless modes in their description, i.e. hydrodynamic fields $\mathscr{B}$ are the only relevant dynamical degrees of freedom at low energies. They also do not have an independent spin current and are coupled to spacetime backgrounds without torsion. We already talked about them in section 1.1.2 using the conventional formalism of hydrodynamics. Here we rederive these results using the off-shell formalism of hydrodynamics proposed in chapter 2. The results we obtain here are standard in the literature; see e.g. [52] for a modern review.

3.1.1 Ideal fluids

Hydrodynamic systems whose constitutive relations are truncated to the zeroth order in derivatives are known as *ideal fluids*. With our choice of derivative counting it is easy to see that all except Class $\mathrm{H_S}$ constitutive relations start at the one-derivative order, so we can safely ignore them for now. At the ideal order, the hydrostatic scalar density $\mathcal{N}$ characterising Class $\mathrm{H_S}$ is given by an arbitrary scalar $P(T, \mu)$ as a function of the temperature T and chemical potential μ of the fluid. Using the definitions of T and μ in terms of $\mathscr{B}$ given in eq. (2.42), we can derive their $\delta_\mathscr{B}$ variations as

$$\delta_\mathscr{B} T = \frac{T}{2} u^\mu u^\nu \delta_\mathscr{B} g_{\mu\nu}, \qquad \delta_\mathscr{B} \frac{\mu}{T} = \frac{1}{T} u^\mu \delta_\mathscr{B} A_\mu. \tag{3.1}$$

We can use these expressions to compute the divergence of the ideal order free energy current $P\beta^\mu$ leading to

$$\begin{aligned} \mathrm{D}_\mu \left(P\beta^\mu\right) &= \frac{1}{\sqrt{-g}} \delta_\mathscr{B} \left(\sqrt{-g}\, P\right) = \frac{1}{2} P g^{\mu\nu} \delta_\mathscr{B} g_{\mu\nu} + \frac{\partial P}{\partial T} \delta_\mathscr{B} T + \frac{\partial P}{\partial \mu} \cdot \delta_\mathscr{B} \mu \\ &= \Big((E + P)\, u^\mu u^\nu + P\, g^{\mu\nu} \Big) \frac{1}{2} \delta_\mathscr{B} g_{\mu\nu} + Q u^\mu \cdot \delta_\mathscr{B} A_\mu, \end{aligned} \tag{3.2}$$

where we have defined

$$S = \frac{\partial P}{\partial T}, \qquad Q = \frac{\partial P}{\partial \mu}, \qquad E = T\frac{\partial P}{\partial T} + \mu \cdot \frac{\partial P}{\partial \mu} - P. \tag{3.3}$$

Comparing eq. (3.2) with eq. (2.74), we can easily read out the ideal fluid constitutive relations, free energy, and entropy currents

$$\begin{aligned} T^{\mu\nu} &= (E+P)\, u^\mu u^\nu + P\, g^{\mu\nu} + \mathcal{O}(\partial), \qquad J^\mu = Q u^\mu + \mathcal{O}(\partial), \\ N^\mu &= \frac{1}{T} P u^\mu + \mathcal{O}(\partial), \qquad J^\mu_{\mathrm{S}} = S u^\mu + \mathcal{O}(\partial). \end{aligned} \tag{3.4}$$

From here we can identify E as the energy density, P as the isotropic pressure, Q as the charge density, and S as the entropy density of the fluid. Due to eq. (3.3), these quantities satisfy the standard thermodynamic relations

$$\begin{aligned} &\text{Gibbs-Duhem equation:} && \mathrm{d}P = S\,\mathrm{d}T + Q\cdot \mathrm{d}\mu, \\ &\text{Euler scaling relation:} && E + P = S\,T + Q\cdot\mu, \\ &\text{First law of thermodynamics:} && \mathrm{d}E = T\,\mathrm{d}S + \mu\cdot\mathrm{d}Q. \end{aligned} \tag{3.5}$$

We see that the constitutive relations of an ideal fluid are completely characterised by its *equation of state* $P = P(T, \mu)$.

We can also work out the explicit first order equations of motion

$$\begin{aligned} \frac{1}{\sqrt{-g}}\delta_{\mathcal{B}}\left(\sqrt{-g}\, T\,(E+P)\, u_\mu\right) + TQ\cdot\delta_{\mathcal{B}}A_\mu + \mathcal{O}(\partial^2) = 0, \\ \frac{1}{\sqrt{-g}}\delta_{\mathcal{B}}\left(\sqrt{-g}\, TQ\right) + \mathcal{O}(\partial^2) = 0, \end{aligned} \tag{3.6}$$

which provide dynamics for β^μ and Λ_β respectively. In this form, it is clear that the hydrostatic principle is satisfied, i.e. upon promoting $\mathcal{B}$ to a background isometry $\mathcal{K}$, the equations of motion are trivially satisfied. It is also clear that we can use these equations to eliminate $u^\mu\delta g_{\mu\nu}$ and $u^\mu\delta A_\mu$ respectively from our non-hydrostatic constitutive relations. In a more readable form, these equations imply

$$\begin{aligned} \mathrm{D}_\mu\left(S u^\mu\right) = \mathcal{O}(\partial^2), \qquad \mathrm{D}_\mu\left(Q u^\mu\right) = \mathcal{O}(\partial^2), \\ (E+P)P^{\mu\nu}\left(\frac{1}{T}\partial_\nu T + u^\rho \mathrm{D}_\rho u_\nu\right) + QP^{\mu\nu}\left(T\mathrm{D}_\nu\frac{\mu}{T} + u^\rho F_{\rho\nu}\right) = \mathcal{O}(\partial^2), \end{aligned} \tag{3.7}$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projector against u^μ. They can be identified as the ideal order entropy conservation, charge continuity equation, and the relativistic Navier-Stokes equation respectively. We see that the ideal fluid indeed satisfies the second law of thermodynamics.

3.1.2 One-derivative corrections

Having discussed the constitutive relations of an ideal fluid, we would now like to explore the most generic one-derivative corrections they can admit, allowed by the second law of

$\boldsymbol{\mathcal{P}}_{\mathsf{T}}$	$T^{\mu\nu}$	J^μ	N^μ
$C\,\boldsymbol{F}\wedge\boldsymbol{F}\wedge\boldsymbol{F}$	$2\mu^2 C\, u^{(\mu}\left(3B^{\nu)} + 2\mu\omega^{\nu)}\right)$	$3\mu C\left(2B^\mu + \mu\omega^\mu\right)$	$\frac{\mu^2}{T}C\left(3B^\mu + \mu\omega^\mu\right)$
$C_1\,\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}_{\mathsf{T}}$	$4T^3 C_1 u^{(\mu}\omega^{\nu)}$		$4T^2 C_1\omega^\mu$
$C_2\,\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}$	$2T^2 C_2 u^{(\mu}\left(B^{\nu)} + 2\mu\omega^{\nu)}\right)$	$T^2 C_2\omega^\mu$	$TC_2\left(B^\mu + 3\mu\omega^\mu\right)$
$C_0\,\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}\wedge\boldsymbol{F}$	$4\mu T C_0 u^{(\mu}\left(B^{\nu)} + \mu\omega^{\nu)}\right)$	$2TC_0\left(B^\mu + \mu\omega^\mu\right)$	$\mu C_0\left(2B^\mu + \mu\omega^\mu\right)$ $+\frac{1}{2}C_0\,\epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}$ $-\frac{1}{3}C_0\,\epsilon^{\mu\nu\rho\sigma}A_\nu A_\rho A_\sigma$

Table 3.1: One-derivative Class A and Class $\mathrm{H_V}$ constitutive relations for a (3+1)-dimensional relativistic fluid. Lie algebra traces are understood in columns $\boldsymbol{\mathcal{P}}_{\mathsf{T}}$, $T^{\mu\nu}$, and N^μ. Note that the C_0 term is in the thermal anomaly polynomial is linear in $\boldsymbol{F}_{\mathsf{T}}$, hence the associated free energy current is not gauge-invariant.

$\mathfrak{D}_0$	$T^{\mu\nu}$	J^μ
$-T\zeta\begin{pmatrix} P^{\mu\nu}P^{\rho\sigma} & \cdot \\ \cdot & \cdot \end{pmatrix}$	$-\zeta\,P^{\mu\nu}\Theta$	
$-T\eta\begin{pmatrix} P^{\mu\langle\rho}P^{\sigma\rangle\nu} & \cdot \\ \cdot & \cdot \end{pmatrix}$	$-\eta\,\sigma^{\mu\nu}$	
$-T\sigma\begin{pmatrix} \cdot & \cdot \\ \cdot & P^{\mu\rho} \end{pmatrix}$		$-\sigma P^{\mu\nu}\left(T\mathrm{D}_\nu\frac{\mu}{T} - E_\nu\right)$

Table 3.2: One-derivative Class D constitutive relations for a $(3+1)$-dimensional relativistic fluid.

thermodynamics. First, some standard definitions:

$$\Theta = \mathrm{D}_\mu u^\mu, \qquad \sigma^{\mu\nu} = P^{\mu\langle\rho}P^{\sigma\rangle\nu}\mathrm{D}_\rho u_\sigma = P^{\mu\rho}P^{\nu\sigma}\left(\mathrm{D}_{(\mu}u_{\nu)} - \frac{1}{3}P_{\mu\nu}\Theta\right),$$
$$\mathfrak{a}^\mu = u^\nu\mathrm{D}_\nu u^\mu, \qquad \omega^\mu = \epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho u_\sigma, \qquad E^\mu = F^{\mu\nu}u_\nu, \qquad B^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma}. \tag{3.8}$$

In words, they are the expansion, shear, acceleration, and vorticity of the fluid, along with the electric and magnetic fields defined in the rest frame of the fluid. Angular brackets denote a traceless symmetric combination.

Let us start with the hydrostatic sector. We can check that there are no one-derivative scalars to make up $\mathcal{N}$, so Class $\mathrm{H_S}$ is empty. There are however non-trivial Class A and Class $\mathrm{H_V}$ constitutive relations at one-derivative order. They are collectively characterised by a 6-rank thermal anomaly polynomial

$$\boldsymbol{\mathcal{P}}_{\mathsf{T}} = C\ \mathrm{tr}[\boldsymbol{F}\wedge\boldsymbol{F}\wedge\boldsymbol{F}] + C_1\,\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}_{\mathsf{T}} + C_2\,\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\mathrm{tr}\,\boldsymbol{F} + C_0\,\boldsymbol{F}_{\mathsf{T}}\wedge\mathrm{tr}[\boldsymbol{F}\wedge\boldsymbol{F}]. \tag{3.9}$$

C is the standard $\mathrm{U}(1)^3$ anomaly coefficient, while C_1 and C_2 are arbitrary constants. Following our discussion in sections 2.3.2 and 2.3.3, we can easily read out the respective constitutive relations. The results have been summarised in table 3.1.

Next, let us move on to the non-hydrostatic constitutive relations. Since the operator "$\delta_{\mathscr{B}}$" already contains a derivative in its definition, at one-derivative order the only relevant non-hydrostatic constitutive relations are parametrised by $\mathfrak{C}_0$. If we split $\mathfrak{C}_0$ into $\mathfrak{D}_0$ and $\overline{\mathfrak{D}}_0$ according to eq. (2.80), we can check that these are its symmetric and anti-symmetric parts respectively. Given the tensor structures at hand, we can check that there are no available terms that we can use to construct an antisymmetric matrix $\overline{\mathfrak{D}}_0$. Consequently, Class $\overline{\mathrm{D}}$ is empty. There are, however, three possible terms that we can write down in $\mathfrak{D}_0$, leading to the non-trivial Class D constitutive relations. The results have been summarised in table 3.2. The associated quadratic form Δ is given by

$$T\Delta = \zeta\Theta^2 + \eta\,\sigma^{\mu\nu}\sigma_{\mu\nu} + \sigma P^{\mu\nu}\left(T\mathrm{D}_\mu\frac{\mu}{T} - E_\mu\right)\left(T\mathrm{D}_\nu\frac{\mu}{T} - E_\nu\right). \tag{3.10}$$

Since each tensor structure on the right hand side is manifestly positive definite, the second law requirement merely states that the dissipative transport coefficients ζ, η, and σ are all non-negative. They are identified as the bulk viscosity, shear viscosity, and electric conductivity of the fluid respectively.

To summarise, the constitutive relations of a relativistic fluid up to the first order in derivatives are given as

$$\begin{aligned}
T^{\mu\nu} &= (E+P)\,u^\mu u^\nu + P\,g^{\mu\nu} - \zeta\,P^{\mu\nu}\Theta - \eta\,\sigma^{\mu\nu} + 2\left(3\mu^2 C + 2\mu T C_0 + C_2 T^2\right)u^{(\mu}B^{\nu)} \\
&\qquad + 4\left(\mu^3 C + \mu^2 T C_0 + T^3 C_1 + C_2\mu T^2\right)u^{(\mu}\omega^{\nu)} + \mathcal{O}(\partial^2), \\
J^\mu &= Q u^\mu - \sigma P^{\mu\nu}\left(T\mathrm{D}_\nu\frac{\mu}{T} - E_\nu\right) \\
&\qquad + \left(6\mu C + 2TC_0\right)B^\mu + \left(3\mu^2 C + 2\mu T C_0 + C_2 T^2\right)\omega^\mu + \mathcal{O}(\partial^2).
\end{aligned} \tag{3.11}$$

They satisfy the second law of thermodynamics with the free energy and entropy currents

$$\begin{aligned}
N^\mu &= \frac{1}{T}P u^\mu + \frac{1}{T}\left(3\mu^2 C + 2\mu T C_0 + T^2 C_2\right)B^\mu + \frac{1}{T}\left(\mu^3 C + \mu^2 T C_0 + 4T^3 C_1 + 3\mu T^2 C_2\right)\omega^\mu \\
&\qquad + C_0\,\epsilon^{\mu\nu\rho\sigma}\left(\frac{1}{2}A_\nu F_{\rho\sigma} - \frac{1}{3}A_\nu A_\rho A_\sigma\right) + \mathcal{O}(\partial^2), \\
J^\mu_{\mathrm{S}} &= S u^\mu + \frac{\mu}{T}\sigma P^{\mu\nu}\left(T\mathrm{D}_\nu\frac{\mu}{T} - E_\nu\right) + 2\left(\mu C_0 + T C_2\right)B^\mu + \left(\mu^2 C_0 + 3T^2 C_1 + 2\mu T C_2\right)\omega^\mu \\
&\qquad + C_0\,\epsilon^{\mu\nu\rho\sigma}\left(\frac{1}{2}A_\nu F_{\rho\sigma} - \frac{1}{3}A_\nu A_\rho A_\sigma\right) + \mathcal{O}(\partial^2),
\end{aligned} \tag{3.12}$$

provided that the zeroth order transport coefficients are derived from a single function $P(T,\mu)$ via the thermodynamic relations (3.5) and the first order transport coefficients η, ζ, and σ are non-negative.

It should be appreciated that, a priori, the constitutive relations could admit many more tensor structures: 3 at the ideal order and 7 at the one-derivative order after modding out field redefinition and on-shell equivalence, each coming with their own transport coefficients. However, the second law of thermodynamics fixed these 10 coefficients in terms of just 1 arbitrary function, 3 arbitrary non-negative functions, and 4 constants. For example, there could be independent terms in the charge current: $\kappa P^{\mu\nu}\partial_\nu T$ and σE^μ, interpreted as thermal and electric conductivities respectively. Instead, the second law fixes the thermal conductivity to be $\kappa = \mu\sigma/T$.

Before moving on to the next example, we should note that we have derived the fluid constitutive relations (3.11) in a very specific hydrodynamic frame, fixed by requiring the off-shell second law to hold. We can always transform to our preferred frame of choice by performing a field redefinition $u^\mu \to u^\mu + \delta u^\mu$, $T \to T + \delta T$, $\mu \to \mu + \delta\mu$ where $u_\mu \delta u^\mu = 0$. For example, one quite ubiquitous frame in relativistic hydrodynamics is the so called *Landau frame* defined as $T^{\mu\nu}_{\text{Landau}} u_\nu = -E u^\mu$ and $J^\mu_{\text{Landau}} u_\mu = Q u^\mu$. This corresponds to the field redefinition $u^\mu \to u^\mu - \frac{1}{E+P} P^\mu{}_\rho T^{\rho\sigma} u_\sigma$ in eq. (3.11), leading to

$$\begin{aligned} T^{\mu\nu}_{\text{Landau}} &= (E+P)\, u^\mu u^\nu + P\, g^{\mu\nu} - \zeta\, P^{\mu\nu}\Theta - \eta\, \sigma^{\mu\nu} + \mathcal{O}(\partial^2), \\ J^\mu_{\text{Landau}} &= Q u^\mu - \sigma P^{\mu\nu}\left(T \mathrm{D}_\nu \frac{\mu}{T} - E_\nu\right) + \xi_B B^\mu + \xi_\omega \omega^\mu + \mathcal{O}(\partial^2), \end{aligned} \tag{3.13}$$

where we have defined the parity-odd conductivities

$$\begin{aligned} \xi_B &= (6\mu C + 2TC_0) - \frac{Q}{E+P}\left(3\mu^2 C + 2\mu T C_0 + C_2 T^2\right) \\ \xi_\omega &= \left(3\mu^2 C + 2\mu T C_0 + C_2 T^2\right) - \frac{2Q}{E+P}\left(\mu^3 C + \mu^2 T C_0 + T^3 C_1 + C_2 \mu T^2\right). \end{aligned} \tag{3.14}$$

This is perhaps the better known form of the one-derivative order relativistic fluid constitutive relations, which were first derived in full generality in [50].

3.2 Relativistic superfluids

For our next example, we study relativistic superfluids. They describe the hydrodynamic regime of a quantum field theory with a spontaneously broken internal symmetry. The corresponding Goldstone modes act as gapless degrees of freedom in the hydrodynamic description, besides u^μ, T, and μ, providing us with an example of non-trivial φ^i fields. In general, we could break an arbitrary Lie group G of internal symmetries to any Lie subgroup H, but for now we content ourselves with breaking $G = \mathrm{U}(1)$ to $H = \{1\}$. This leads to Abelian superfluids, which we briefly covered in section 1.2. We return to the non-Abelian superfluids in section 3.2.4. Constitutive relations for an Abelian superfluid have already been obtained in the literature [51, 53, 96] using the conventional "on-shell" formalism. Here we employ the off-shell formalism of hydrodynamics discussed in chapter 2 to rederive these results. One important distinction between the two formalisms is that in the conventional formalism, one typically imposes an equation of motion for the Goldstone mode, known as the Josephson equation, by hand. However, in the off-shell formalism, this equation automatically pops out of the second law. The results presented here appeared in our works [1, 4].

3.2.1 Goldstone modes and Josephson equation

Consider a microscopic theory which is charged under a global U(1) symmetry. We are seeking to describe low energy fluctuations in this theory in a phase where the U(1) symmetry is spontaneously broken, perhaps due to condensation of a charged scalar operator. The U(1) phase φ of this operator becomes an additional field in the fluid description, upon which the respective constitutive relations can depend. Under an infinitesimal symmetry transformation parametrised by $\mathcal{X} = (\chi^\mu, \Lambda_\chi)$, the field φ transforms as $\delta_{\mathcal{X}}\varphi = \chi^\mu \partial_\mu \varphi - \Lambda_\chi$.

We can write down a covariant derivative of φ as

$$\xi_\mu = \partial_\mu \varphi + A_\mu, \tag{3.15}$$

commonly known as the *superfluid velocity.* It satisfies $2\partial_{[\mu}\xi_{\nu]} = F_{\mu\nu}$. In superfluid dynamics, it is customary to choose the derivative order of φ to be $\mathcal{O}(\partial^{-1})$, which would imply that ξ_μ is $\mathcal{O}(\partial^0)$. The phase field φ provides us with our first example of an additional gapless mode "φ^i" discussed in section 2.2.3. For the corresponding equation of motion "$\mathcal{E}_i \approx 0$", we take

$$K \approx 0. \tag{3.16}$$

Just like $T^{\mu\nu}$ and J^μ, the placeholder K is also provided with constitutive relations. Its form is fixed by requiring the second law of thermodynamics to hold.

Following eq. (2.53), we can directly write down the adiabaticity equation for superfluids, leading to

$$\mathrm{D}_\mu N^\mu - \mathrm{N}_{\mathrm{H}}^{\perp} = \frac{1}{2} T^{\mu\nu} \delta_{\mathscr{B}} g_{\mu\nu} + J^\mu \delta_{\mathscr{B}} A_\mu + K \delta_{\mathscr{B}} \varphi + \Delta, \qquad \Delta \geq 0, \tag{3.17}$$

where

$$\delta_{\mathscr{B}} \varphi = \beta^\mu \partial_\mu \varphi - \Lambda_\beta = \frac{1}{T}\left(u^\mu \xi_\mu - \mu\right). \tag{3.18}$$

Let us start by considering the adiabaticity equation at the zeroth order in derivatives. Ignoring all the derivatives, eq. (3.17) simply becomes: $-K\delta_{\mathscr{B}}\varphi + \mathcal{O}(\partial) = \Delta \geq 0$. The placeholder K, at this order, is just a function of all the ideal order scalars T, μ, $\mu_s = -\frac{1}{2}\xi^\mu\xi_\mu$, and $\delta_{\mathscr{B}}\varphi$, such that $-K\delta_{\mathscr{B}}\varphi$ is a positive semi-definite quadratic form. It follows that K must take the form

$$K = -\alpha \delta_{\mathscr{B}} \varphi + \mathcal{O}(\partial), \qquad \Delta = \alpha (\delta_{\mathscr{B}} \varphi)^2 + \mathcal{O}(\partial), \tag{3.19}$$

for some transport coefficient $\alpha \geq 0$. Using the φ equation of motion $K \approx 0$, it follows that

$$K = -\alpha \delta_{\mathscr{B}} \varphi = -\frac{\alpha}{T}\left(u^\mu \xi_\mu - \mu\right) \approx \mathcal{O}(\partial) \quad \Longrightarrow \quad u^\mu \xi_\mu \approx \mu + \mathcal{O}(\partial). \tag{3.20}$$

This is known as the *Josephson equation.* Due to this relation, $u^\mu\xi_\mu$ is not independent on-shell and hence is not used as an independent scalar while writing down the superfluid constitutive relations.

3.2.2 Ideal superfluids

Similar to the ordinary ideal fluids, Class $\mathrm{H_S}$ is the only non-empty class for ideal superfluids as well. However, the hydrostatic scalar density $\mathcal{N}$ characterising Class $\mathrm{H_S}$ is given by a three-variable function $P(T, \mu, \mu_s)$ this time, where T is the temperature, μ is the chemical potential, while $\mu_s = -\frac{1}{2}\xi^\mu\xi_\mu$ is the superfluid potential. We have omitted the only other possible ideal order scalar $u^\mu\xi_\mu$ in the functional dependence, because it is not independent on-shell. The $\delta_{\mathscr{B}}$ variations of T and μ are given in eq. (3.1), while for μ_s we find

$$\delta_{\mathscr{B}}\mu_s = \frac{1}{2}\xi^\mu\xi^\nu\delta_{\mathscr{B}}g_{\mu\nu} - \xi^\mu\delta_{\mathscr{B}}A_\mu - \xi^\mu \mathrm{D}_\mu\delta_{\mathscr{B}}\varphi. \tag{3.21}$$

We can use it to compute the divergence

$$\begin{aligned}
\mathrm{D}_\mu\left(\beta^\mu P\right) &= \frac{1}{\sqrt{-g}}\delta_{\mathscr{B}}\left(\sqrt{-g}P\right) = \frac{1}{2}Pg^{\mu\nu}\delta_{\mathscr{B}}g_{\mu\nu} + \frac{\partial P}{\partial T}\delta_{\mathscr{B}}T + \frac{\partial P}{\partial\mu}\delta_{\mathscr{B}}\mu + \frac{\partial P}{\partial\mu_s}\delta_{\mathscr{B}}\mu_s, \\
&= \Big((E+P)u^\mu u^\nu + Pg^{\mu\nu} + R_s\xi^\mu\xi^\nu\Big)\frac{1}{2}\delta_{\mathscr{B}}g_{\mu\nu} + \left(Qu^\mu - R_s\xi^\mu\right)\delta_{\mathscr{B}}A_\mu \\
&\quad + \mathrm{D}_\mu\left(R_s\xi^\mu\right)\delta_{\mathscr{B}}\varphi - \mathrm{D}_\mu\left(R_s\xi^\mu\delta_{\mathscr{B}}\varphi\right),
\end{aligned} \tag{3.22}$$

where we have defined

$$S = \frac{\partial P}{\partial T}, \qquad Q = \frac{\partial P}{\partial\mu}, \qquad R_s = \frac{\partial P}{\partial\mu_s}, \qquad E = T\frac{\partial P}{\partial T} + \mu\frac{\partial P}{\partial\mu} - P. \tag{3.23}$$

Comparing eq. (3.22) with eq. (2.74), we can read out the ideal superfluid constitutive relations, free energy, and entropy currents

$$\begin{aligned}
T^{\mu\nu} &= (E+P)u^\mu u^\nu + Pg^{\mu\nu} + R_s\xi^\mu\xi^\nu + \mathcal{O}(\partial), \\
J^\mu &= Qu^\mu - R_s\xi^\mu + \mathcal{O}(\partial), \\
K &= -\alpha\delta_{\mathscr{B}}\varphi + \mathrm{D}_\mu(R_s\xi^\mu) + \mathcal{O}(\partial), \\
N^\mu &= \frac{1}{T}Pu^\mu + \delta_{\mathscr{B}}\varphi R_s\xi^\mu + \mathcal{O}(\partial), \\
J^\mu_{\mathrm{S}} &= N^\mu - \frac{1}{T}\left(T^{\mu\nu}u_\nu + \mu J^\mu\right) = Su^\mu + \mathcal{O}(\partial).
\end{aligned} \tag{3.24}$$

Like before, P can be identified as the isotropic pressure, while E, Q, and S can be identified as the energy, charge, and entropy densities of the superfluid respectively. The quantity R_s on the other hand is known as the superfluid density. They follow a set of thermodynamic relations

$$\begin{aligned}
\text{Gibbs-Duhem equation:} &\quad \mathrm{d}P = S\mathrm{d}T + Q\mathrm{d}\mu + R_s\mathrm{d}\mu_s, \\
\text{Euler scaling relation:} &\quad E + P = ST + Q\mu, \\
\text{First law of thermodynamics:} &\quad \mathrm{d}E = T\mathrm{d}S + \mu\mathrm{d}Q - R_s\mathrm{d}\mu_s,
\end{aligned} \tag{3.25}$$

which are directly implied by eq. (3.23). We see that an ideal superfluid is also completely characterised by its equation of state $P = P(T,\mu,\mu_s)$. Note that we have included some one-derivative terms in K and N^μ which can be ignored when working at ideal order, but are required for the internal consistency with the second law.

Let us work out the first order equations of motion. By a direct computation we get

$$\begin{aligned}
\frac{1}{\sqrt{-g}}\delta_{\mathscr{B}}\left(\sqrt{-g}\,T(E+P)u_\mu\right) + QT\delta_{\mathscr{B}}A_\mu + \xi_\nu \mathrm{D}_\mu(R_s\xi^\mu) + \mathcal{O}(\partial^2) &= 0, \\
\frac{1}{\sqrt{-g}}\delta_{\mathscr{B}}\left(\sqrt{-g}\,QT\right) - \mathrm{D}_\mu(R_s\xi^\mu) + \mathcal{O}(\partial^2) &= 0, \\
\alpha\delta_{\mathscr{B}}\varphi - \mathrm{D}_\mu(R_s\xi^\mu) + \mathcal{O}(\partial) &= 0.
\end{aligned} \tag{3.26}$$

They provide dynamics to β^μ, Λ_β, and φ respectively. In a hydrostatic configuration, all these equations boil down to a scalar equation $\mathrm{D}_\mu(R_s\xi^\mu) = 0$, which determines the equilibrium profile of the phase field φ. Substituting the respective expressions for $\delta_{\mathscr{B}}$ variations, these equations imply

$$\begin{aligned}
\mathrm{D}_\mu\left(Su^\mu\right) &= \mathcal{O}(\partial^2), \\
\mathrm{D}_\mu\left(Qu^\mu\right) &= \mathrm{D}_\mu(R_s\xi^\mu) + \mathcal{O}(\partial^2), \\
(E+P)\zeta^\mu\left(\frac{1}{T}\partial_\mu T + u^\rho \mathrm{D}_\rho u_\mu\right) + Q\zeta^\mu\left(T\partial_\mu\frac{\mu}{T} + u^\rho F_{\rho\mu}\right) + \zeta^2\mathrm{D}_\nu(R_s\xi^\nu) &= \mathcal{O}(\partial^2), \\
(E+P)P_\zeta^{\mu\nu}\left(\frac{1}{T}\partial_\nu T + u^\rho \mathrm{D}_\rho u_\nu\right) + QP_\zeta^{\mu\nu}\left(T\partial_\nu\frac{\mu}{T} + u^\rho F_{\rho\nu}\right) &= \mathcal{O}(\partial^2), \\
u^\mu\xi_\mu &= \mu + \frac{T}{\alpha}\mathrm{D}_\mu(R_s\xi^\mu) + \mathcal{O}(\partial),
\end{aligned} \tag{3.27}$$

where we have defined $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, $\zeta^\mu = P^{\mu\nu}\xi_\nu$, and $P_\zeta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu - \frac{1}{\zeta^2}\zeta^\mu\zeta^\nu$. Note that $\zeta^2 = \mu^2 - 2\mu_s + \mathcal{O}(\partial)$. The first equation is the ideal order entropy conservation. The second equation serves as the charge continuity equation, balancing the flow of charges along u^μ and ξ^μ. The third and forth equations are components of the relativistic Navier-Stokes equation along and transverse to ζ^μ. Finally, the fifth equation is an improvement of the Josephson equation. Note, however, that this equation can admit further one derivative corrections due to the first order constitutive relations discussed in the next subsection; the correction mentioned here is only how the ideal superfluid transport affects the Josephson equation.

3.2.3 One-derivative corrections

We now improve our superfluid constitutive relations with one-derivative corrections. In table 3.3 we have listed various one-derivative tensor structures that we require in our discussion below.

Let us start with the hydrostatic sector. Class $\mathrm{H_S}$, unlike ordinary fluids, is non-empty for superfluids. Ignoring the total derivative terms, the corresponding hydrostatic scalar density $\mathcal{N}$ at the first order in derivatives is now given as

$$\mathcal{N} = P + f_1\,S_1 + f_2\,S_2 + g_1\,\tilde{S}_{e,1} + g_2\,\tilde{S}_{e,2}. \tag{3.28}$$

We have chosen not to include the first order hydrostatic scalars: $\mathrm{D}_\mu\xi^\mu$ and $\xi^\mu\partial_\mu\mu_s$, as the former is a total derivative and the latter can be exchanged for $\mathrm{D}_\mu(R_s\xi^\mu)$, which is not

Non-hydrostatic — on-shell independent		
S_1	$\frac{T}{2}P_\zeta^{\mu\nu}\delta_{\mathscr{B}}g_{\mu\nu}$	$P^{\mu\nu}\mathrm{D}_\mu u_\nu$
S_2	$\frac{T}{2}\zeta^\mu\zeta^\nu\delta_{\mathscr{B}}g_{\mu\nu}$	$\zeta^\mu\zeta^\nu\mathrm{D}_\mu u_\nu$
S_3	$T\zeta^\mu\delta_{\mathscr{B}}A_\mu$	$\zeta^\mu\left(T\partial_\mu\frac{\mu}{T}-E_\mu\right)$
S_4	$T\delta_{\mathscr{B}}\varphi$	$u^\mu\xi_\mu-\mu=\frac{T}{\alpha}\mathrm{D}_\mu(R_s\xi^\mu)+\mathcal{O}(\partial)$
V_1^μ	$TP_\zeta^{\mu\nu}\zeta^\rho\delta_{\mathscr{B}}g_{\nu\rho}$	$2P_\zeta^{\mu\nu}\zeta^\rho\mathrm{D}_{(\nu}u_{\rho)}$
V_2^μ	$TP_\zeta^{\mu\nu}\delta_{\mathscr{B}}A_\mu$	$P_\zeta^{\mu\nu}\left(T\partial_\nu\frac{\mu}{T}+E_\nu\right)$
$\sigma_\zeta^{\mu\nu}$	$\frac{T}{2}P_\zeta^{\rho\langle\mu}P_\zeta^{\nu\rangle\sigma}\delta_{\mathscr{B}}g_{\rho\sigma}$	$P_\zeta^{\mu\rho}P_\zeta^{\nu\sigma}\left(\mathrm{D}_{(\rho}u_{\sigma)}-\frac{1}{2}P^\zeta_{\rho\sigma}S_1\right)$
$\tilde{V}_1^\mu$	$\epsilon^{\mu\nu\rho\sigma}u_\nu\zeta_\rho V_{1,\sigma}$	
$\tilde{V}_2^\mu$	$\epsilon^{\mu\nu\rho\sigma}u_\nu\zeta_\rho V_{2,\sigma}$	
$\tilde{\sigma}_\zeta^{\mu\nu}$	$P_\zeta^{\lambda(\mu}\epsilon^{\nu)\rho\sigma\tau}u_\rho\zeta_\sigma\sigma^\zeta_{\tau\lambda}$	
Non-hydrostatic — on-shell dependent		
S_5	$\frac{T}{2}u^\mu u^\nu\delta_{\mathscr{B}}g_{\mu\nu}$	$\frac{1}{T}u^\mu\partial_\mu T$
S_6	$Tu^\mu\delta_{\mathscr{B}}A_\mu$	$Tu^\mu\partial_\mu\frac{\mu}{T}$
S_7	$T\zeta^\mu u^\nu\delta_{\mathscr{B}}g_{\mu\nu}$	$\zeta^\nu\left(\frac{1}{T}\partial_\nu T+\mathfrak{a}_\nu\right)$
V_3^μ	$TP_\zeta^{\mu\nu}u^\rho\delta_{\mathscr{B}}g_{\nu\rho}$	$P_\zeta^{\mu\nu}\left(\frac{1}{T}\partial_\nu T+\mathfrak{a}_\nu\right)$
$\tilde{V}_3^\mu$	$\epsilon^{\mu\nu\rho\sigma}u_\nu\zeta_\rho V_{3,\sigma}$	
Hydrostatic		
$S_{e,1}$	$\frac{1}{T}\zeta^\mu\partial_\mu T$	
$S_{e,2}$	$T\zeta^\mu\partial_\mu\frac{\mu}{T}$	
$V_{e,1}^\mu$	$\frac{1}{T}P_\zeta^{\mu\nu}\partial_\nu T$	
$V_{e,2}^\mu$	$TP_\zeta^{\mu\nu}\partial_\nu\nu$	
$\tilde{S}_{e,1}$	$T\epsilon^{\mu\nu\rho\sigma}\zeta_\mu u_\nu\partial_\rho u_\sigma$	
$\tilde{S}_{e,2}$	$\frac{1}{2}T\epsilon^{\mu\nu\rho\sigma}\zeta_\mu u_\nu F_{\rho\sigma}$	
$\tilde{V}_{e,1}^\mu$	$TP_\zeta{}^\mu{}_\tau\epsilon^{\tau\nu\rho\sigma}u_\nu\partial_\rho u_\sigma$	
$\tilde{V}_{e,2}^\mu$	$\frac{1}{2}TP_\zeta{}^\mu{}_\tau\epsilon^{\tau\nu\rho\sigma}u_\nu F_{\rho\sigma}$	
$\tilde{V}_{e,3}^\mu$	$TP_\zeta{}^\mu{}_\tau\epsilon^{\tau\nu\rho\sigma}\xi_\nu\partial_\rho u_\sigma$	
$\tilde{V}_{e,4}^\mu$	$\frac{1}{2}TP_\zeta{}^\mu{}_\tau\epsilon^{\tau\nu\rho\sigma}\xi_\nu F_{\rho\sigma}$	
$\vdots$	$\vdots$	

Table 3.3: Independent first order data for $(3+1)$-dimensional relativistic superfluids. We have not enlisted, neither would we need, all the independent data surviving at equilibrium.

$\mathcal{N}$	$T^{\mu\nu}$	J^μ
$f_1\, S_{e,1}$	$\left(\alpha_{E,1}u^\mu u^\nu + \alpha_{R_s,1}(\zeta^\mu\zeta^\nu - 2\mu u^{(\mu}\zeta^{\nu)}) + f_1 P_\zeta^{\mu\nu}\right) S_{e,1}$ $-2f_1\xi^{(\mu}V_{e,1}^{\nu)} + 2f_1 u^{(\mu}\zeta^{\nu)}S_5 - u^\mu u^\nu \frac{1}{T}\mathrm{D}_\sigma\left(Tf_1\zeta^\sigma\right)$	$\left(\alpha_{Q,1}u^\mu - \alpha_{R_s,1}\zeta^\mu\right) S_{e,1}$ $+f_1 V_{e,1}^\mu$
$f_2\, S_{e,2}$	$\left(\alpha_{E,2}u^\mu u^\nu + \alpha_{R_s,2}(\zeta^\mu\zeta^\nu - 2\mu u^{(\mu}\zeta^{\nu)}) + f_2 P_\zeta^{\mu\nu}\right) S_{e,2}$ $-2f_2\xi^{(\mu}V_{e,2}^{\nu)} + 2u^{(\mu}\zeta^{\nu)}f_2 S_6$	$\left(\alpha_{Q,2}u^\mu - \alpha_{R_s,2}\zeta^\mu\right) S_{e,2}$ $+f_2 V_{e,2}^\mu$ $-u^\mu \frac{1}{T}\mathrm{D}_\nu(Tf_2\zeta^\nu)$
$g_1\, \tilde{S}_{e,1}$	$\left(\tilde{\alpha}_{E,1}u^\mu u^\nu + \tilde{\alpha}_{R_s,1}(\zeta^\mu\zeta^\nu - 2\mu u^{(\mu}\zeta^{\nu)})\right)\tilde{S}_{e,1}$ $+\left(g_1 u^\mu u^\nu + \frac{g_1}{\zeta^2}\zeta^\mu\zeta^\nu\right)\tilde{S}_{e,1}$ $-2g_1 u^{(\mu}\tilde{V}_{e,3}^{\nu)} - 2u^{(\mu}\epsilon^{\nu)\rho\sigma\tau}\mathrm{D}_\sigma\left(Tg_1 u_\tau\zeta_\rho\right)$	$\left(\tilde{\alpha}_{Q,1}u^\mu - \tilde{\alpha}_{R_s,1}\zeta^\mu\right)\tilde{S}_{e,1}$ $+g_1\tilde{V}_{e,1}^\mu$
$g_2\, \tilde{S}_{e,2}$	$\left(\tilde{\alpha}_{E,2}u^\mu u^\nu + \tilde{\alpha}_{R_s,2}(\zeta^\mu\zeta^\nu - 2\mu u^{(\mu}\zeta^{\nu)})\right)\tilde{S}_{e,2}$ $+\frac{g_1}{\zeta^2}\zeta^\mu\zeta^\nu\tilde{S}_{e,1} - 2u^{(\mu}g_2\tilde{V}_{e,4}^{\nu)}$	$\left(\tilde{\alpha}_{Q,2}u^\mu - \tilde{\alpha}_{R_s,2}\zeta^\mu\right)\tilde{S}_{e,2}$ $+g_2\tilde{V}_{e,2}^\mu$ $+\epsilon^{\mu\nu\rho\sigma}\mathrm{D}_\nu(Tg_2\zeta_\rho u_\sigma)$

$\mathcal{N}$	K	N^μ
$f_1\, S_{e,1}$	$\mathrm{D}_\mu\left(\zeta^\mu\alpha_{R_s,1}S_{e,1} - f_1 V_{e,1}^\mu\right)$	$\frac{1}{T}f_1\left(u^\mu S_{e,1} - \zeta^\mu S_5\right)$
$f_2\, S_{e,2}$	$\mathrm{D}_\mu\left(\zeta^\mu\alpha_{R_s,2}S_{e,1} - f_1 V_{e,2}^\mu\right)$	$\frac{1}{T}f_2\left(u^\mu S_{e,2} - \zeta^\mu S_6\right)$
$g_1\, \tilde{S}_{e,1}$	$\mathrm{D}_\mu\left(\zeta^\mu\tilde{\alpha}_{R_s,1}\tilde{S}_{e,1} - g_1\tilde{V}_{e,1}^\mu\right)$	$g_1\left(\frac{1}{T}u^\mu\tilde{S}_{e,1} + \tilde{V}_3^\mu\right)$
$g_2\, \tilde{S}_{e,2}$	$\mathrm{D}_\mu\left(\zeta^\mu\tilde{\alpha}_{R_s,2}\tilde{S}_{e,2} - g_2\tilde{V}_{e,2}^\mu\right)$	$g_2\left(\frac{1}{T}u^\mu\tilde{S}_{e,2} + \tilde{V}_2^\mu\right)$

Table 3.4: One-derivative Class $\mathrm{H_S}$ constitutive relations for a $(3+1)$-dimensional relativistic superfluid.

hydrostatic due to the Josephson equation. For clarity, let us define the derivatives of the transport coefficients appearing above as

$$\begin{aligned} \mathrm{d}f_i &= \frac{\alpha_{E,i}}{T}\mathrm{d}T + T\alpha_{Q,i}\mathrm{d}\frac{\mu}{T} - \frac{1}{2}\left(\alpha_{R_s,i} + \frac{f_i}{\zeta^2}\right)\mathrm{d}\zeta^2, \\ \mathrm{d}g_i &= \frac{\tilde{\alpha}_{E,i}}{T}\mathrm{d}T + T\tilde{\alpha}_{Q,i}\mathrm{d}\frac{\mu}{T} - \frac{1}{2}\left(\tilde{\alpha}_{R_s,i} + \frac{g_i}{\zeta^2}\right)\mathrm{d}\zeta^2, \end{aligned} \tag{3.29}$$

along with

$$\alpha_{E,i} + f_i = \alpha_{S,i}T + \alpha_{Q,i}\mu, \qquad \tilde{\alpha}_{E,i} + g_i = \tilde{\alpha}_{S,i}T + \tilde{\alpha}_{Q,i}\mu. \tag{3.30}$$

Comparing the variation of $\mathcal{N}$ to eq. (2.74) and using these definitions, we can read out the Class $\mathrm{H_S}$ constitutive relations. The algebra is quite involved; see the appendix of [4] for detailed steps. For the benefit of the reader, we have summarised the results in table 3.4. Moving on, introduction of the phase field φ does not alter the (thermal) anomaly polynomial in any way, therefore Class A and Class $\mathrm{H_V}$ results can be directly imported from table 3.1. The only worthwhile comment is that we do not need to consider C_0 and C_2 terms in Class $\mathrm{H_V}$ independently. Due to the presence of a "gauge fixed" version of the gauge field $\xi_\mu = A_\mu + \partial_\mu\varphi$, these terms can be absorbed into Class $\mathrm{H_S}$ by shifting $g_1 \to g_1 - \mu C_0 - TC_2$ and $g_2 \to g_2 - C_0$. However, the C_1 term in Class $\mathrm{H_V}$ and C term in Class A remain independent.

$\mathfrak{D}_0$	$T^{\mu\nu}$	J^μ	K
$-T\beta_{11}\begin{pmatrix} P_\zeta^{\mu\nu}P_\zeta^{\rho\sigma} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{11}P_\zeta^{\mu\nu}S_1$		
$-T\beta_{(12)}\begin{pmatrix} P_\zeta^{\mu\nu}\zeta^\rho\zeta^\sigma + \zeta^\mu\zeta^\nu P_\zeta^{\rho\sigma} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{(12)}\left(P_\zeta^{\mu\nu}S_2 + \zeta^\mu\zeta^\nu S_1\right)$		
$-T\beta_{(13)}\begin{pmatrix} \cdot & \zeta^\rho P_\zeta^{\mu\nu} & \cdot \\ \zeta^\mu P_\zeta^{\rho\sigma} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{(13)}P_\zeta^{\mu\nu}S_3$	$-\beta_{(13)}\zeta^\mu S_1$	
$-T\beta_{(14)}\begin{pmatrix} \cdot & \cdot & P_\zeta^{\mu\nu} \\ \cdot & \cdot & \cdot \\ P_\zeta^{\rho\sigma} & \cdot & \cdot \end{pmatrix}$	$-\beta_{(14)}P_\zeta^{\mu\nu}S_4$		$-\beta_{(14)}S_1$
$-T\beta_{22}\begin{pmatrix} \zeta^\mu\zeta^\nu\zeta^\rho\zeta^\sigma & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{22}\zeta^\mu\zeta^\nu S_2$		
$-T\beta_{(23)}\begin{pmatrix} \cdot & \zeta^\rho\zeta^\mu\zeta^\nu & \cdot \\ \zeta^\mu\zeta^\rho\zeta^\sigma & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{(23)}\zeta^\mu\zeta^\nu S_3$	$-\beta_{(23)}\zeta^\mu S_2$	
$-T\beta_{(24)}\begin{pmatrix} \cdot & \cdot & \zeta^\mu\zeta^\nu \\ \cdot & \cdot & \cdot \\ \zeta^\rho\zeta^\sigma & \cdot & \cdot \end{pmatrix}$	$-\beta_{(24)}\zeta^\mu\zeta^\nu S_4$		$-\beta_{(24)}S_2$
$-T\beta_{33}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \zeta^\mu\zeta^\rho & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$		$-\beta_{33}\zeta^\mu S_3$	
$-T\beta_{(34)}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \zeta^\mu \\ \cdot & \zeta^\rho & \cdot \end{pmatrix}$		$-\beta_{(34)}\zeta^\mu S_4$	$-\beta_{(34)}S_3$
$-T\beta_{44}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}$			$-\beta_{44}S_4$
$-4T\kappa_{11}\begin{pmatrix} \zeta^{(\mu}P_\zeta^{\nu)(\rho}\zeta^{\sigma)} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\kappa_{11}\zeta^{(\mu}V_1^{\nu)}$		
$-2T\kappa_{(12)}\begin{pmatrix} \cdot & \zeta^{(\mu}P_\zeta^{\nu)\rho} & \cdot \\ \zeta^{(\rho}P_\zeta^{\sigma)\mu} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\kappa_{(12)}\zeta^{(\mu}V_2^{\nu)}$	$-\kappa_{(12)}V_1^\mu$	
$-T\kappa_{22}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & P_\zeta^{\mu\rho} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$		$-\kappa_{22}V_2^\mu$	
$2T\tilde{\kappa}_{[12]}\begin{pmatrix} \cdot & \zeta^{(\mu}\tilde{\epsilon}^{\nu)\rho} & \cdot \\ -\tilde{\epsilon}^{\mu(\rho}\zeta^{\sigma)} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\tilde{\kappa}_{[12]}\zeta^{(\mu}\tilde{V}_2^{\nu)}$	$\tilde{\kappa}_{[12]}\tilde{V}_1^\mu$	
$-T\eta\begin{pmatrix} P_\zeta^{\mu\langle\rho}P_\zeta^{\sigma\rangle\nu} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\eta\sigma_\zeta^{\mu\nu}$		

Table 3.5: One-derivative Class D constitutive relations for a $(3+1)$-dimensional superfluid. We have defined $\tilde{\epsilon}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}u_\rho\zeta_\sigma$. Note that we have included the dissipative transport coefficient $\beta_{44} = \alpha/T$ in K for completeness.

In the non-hydrostatic sector, we need to write the most generic parametrisation of $\mathfrak{C}_0$ which decomposes into a symmetric part $\mathfrak{D}_0$ and an antisymmetric part $\overline{\mathfrak{D}}_0$. They correspond to Class D and Class $\overline{\mathrm{D}}$ constitutive relations respectively. There are a total of 11 transport coefficients in Class $\overline{\mathrm{D}}$: $[\beta_{[ij]}]_{4\times4}$, $[\kappa_{[ij]}]_{2\times2}$, $[\tilde{\kappa}_{(ij)}]_{2\times2}$, and $\tilde{\eta}$. On the other hand, there are 15 transport coefficients in Class D: $[\beta_{(ij)}]_{4\times4}$, $[\kappa_{(ij)}]_{2\times2}$, $[\tilde{\kappa}_{[ij]}]_{2\times2}$, and η, including $\beta_{44} = \alpha/T$ discussed in section 3.2.1. These results have been presented in tables 3.5 and 3.6. The quadratic form Δ is given by

$$T\Delta = \sum_{i,j=1}^{4} S_i\beta_{(ij)}S_j + \left(\sum_{i,j=1}^{2} V_i^\mu\kappa_{(ij)}V_{i,\mu} + \sum_{i=1}^{2} V_i^\mu\tilde{\kappa}_{[ij]}\tilde{V}_{j,\mu}\right) + \eta\sigma^{\mu\nu}\sigma_{\mu\nu}. \tag{3.31}$$

Note that

$$(\epsilon^{\mu\nu\rho\sigma}u_\rho\zeta_\sigma)(\epsilon_{\tau\nu\alpha\beta}u^\alpha\zeta^\beta) = \zeta^2 P_\zeta{}^\mu{}_\tau. \tag{3.32}$$

$\overline{\mathfrak{D}}_0$	$T^{\mu\nu}$	J^μ	K
$-T\beta_{[12]}\begin{pmatrix} \tilde{P}^{\mu\nu}\zeta^\rho\zeta^\sigma - \zeta^\mu\zeta^\nu\tilde{P}^{\rho\sigma} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{[12]}\left(\tilde{P}^{\mu\nu}S_2 - \zeta^\mu\zeta^\nu S_1\right)$		
$-T\beta_{[13]}\begin{pmatrix} \cdot & \zeta^\rho P_\zeta^{\mu\nu} & \cdot \\ -\zeta^\mu P_\zeta^{\rho\sigma} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{[13]}P_\zeta^{\mu\nu}S_3$	$\beta_{[13]}\zeta^\mu S_1$	
$-T\beta_{[14]}\begin{pmatrix} \cdot & \cdot & P_\zeta^{\mu\nu} \\ \cdot & \cdot & \cdot \\ -P_\zeta^{\rho\sigma} & \cdot & \cdot \end{pmatrix}$	$-\beta_{[14]}P_\zeta^{\mu\nu}S_4$		$\beta_{[14]}S_1$
$-T\beta_{[23]}\begin{pmatrix} \cdot & \zeta^\rho\zeta^\mu\zeta^\nu & \cdot \\ -\zeta^\mu\zeta^\rho\zeta^\sigma & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{[23]}\zeta^\mu\zeta^\nu S_3$	$\beta_{[23]}\zeta^\mu S_2$	
$-T\beta_{[24]}\begin{pmatrix} \cdot & \cdot & \zeta^\mu\zeta^\nu \\ \cdot & \cdot & \cdot \\ -\zeta^\rho\zeta^\sigma & \cdot & \cdot \end{pmatrix}$	$-\beta_{[24]}\zeta^\mu\zeta^\nu S_4$		$\beta_{[24]}S_2$
$-T\beta_{[34]}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \zeta^\mu \\ \cdot & -\zeta^\rho & \cdot \end{pmatrix}$		$-\beta_{[34]}\zeta^\mu S_4$	$\beta_{[34]}S_3$
$-2T\kappa_{[12]}\begin{pmatrix} \cdot & \zeta^{(\mu}P_\zeta^{\nu)\rho} & \cdot \\ -\zeta^{(\rho}P_\zeta^{\sigma)\mu} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\kappa_{[12]}\zeta^{(\mu}V_2^{\nu)}$	$\kappa_{[12]}V_1^\mu$	
$4T\tilde{\kappa}_{11}\begin{pmatrix} \zeta^{(\mu}\tilde{\epsilon}^{\nu)(\rho}\zeta^{\sigma)} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\tilde{\kappa}_{11}\zeta^{(\mu}\tilde{V}_1^{\nu)}$		
$2T\tilde{\kappa}_{(12)}\begin{pmatrix} \cdot & \zeta^{(\mu}\tilde{\epsilon}^{\nu)\rho} & \cdot \\ \tilde{\epsilon}^{\mu(\rho}\zeta^{\sigma)} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\tilde{\kappa}_{(12)}\zeta^{(\mu}\tilde{V}_2^{\nu)}$	$-\tilde{\kappa}_{(12)}\tilde{V}_1^\mu$	
$T\tilde{\kappa}_{22}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \tilde{\epsilon}^{\mu\rho} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$		$-\tilde{\kappa}_{22}\tilde{V}_2^\mu$	
$T\tilde{\eta}\begin{pmatrix} P_\zeta^{\lambda(\mu}\tilde{\epsilon}^{\nu)(\rho}P_\zeta^{\sigma)}{}_\lambda & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\tilde{\eta}\tilde{\sigma}_\zeta^{\mu\nu}$		

Table 3.6: One-derivative Class $\overline{\mathrm{D}}$ constitutive relations for a $(3+1)$-dimensional superfluid. We have defined $\tilde{\epsilon}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}u_\rho\zeta_\sigma$.

We can use this to define a new basis of vector structures

$$\begin{pmatrix} V_1'^\mu \\ V_2'^\mu \end{pmatrix} = \begin{pmatrix} V_1^\mu \\ V_2^\mu \end{pmatrix} + \begin{pmatrix} 0 & \tilde{\kappa}_{12}^{(a)}/\kappa_{11} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \tilde{V}_1^\mu \\ \tilde{V}_2^\mu \end{pmatrix},$$
$$\begin{pmatrix} \kappa'_{11} & \kappa'_{(12)} \\ \kappa'_{(12)} & \kappa'_{22} \end{pmatrix} = \begin{pmatrix} \kappa_{11} & \kappa_{(12)} \\ \kappa_{(12)} & \kappa_{22} - \zeta^2\frac{\tilde{\kappa}_{[12]}}{\kappa_{11}} \end{pmatrix}, \tag{3.33}$$

such that

$$\sum_{i,j=1}^{2} V_i'^\mu\kappa'_{(ij)}V'_{i,\mu} = \sum_{i,j=1}^{2} V_i^\mu\kappa_{(ij)}V_{i,\mu} + \sum_{i=1}^{2} V_i^\mu\tilde{\kappa}_{[ij]}\tilde{V}_{j,\mu}. \tag{3.34}$$

In this basis, Δ takes the form

$$T\Delta = \sum_{i,j=1}^{4} S_i\beta_{(ij)}S_j + \sum_{i,j=1}^{2} V_i'^\mu\kappa'_{(ij)}V'_{i,\mu} + \eta\sigma^{\mu\nu}\sigma_{\mu\nu}. \tag{3.35}$$

Provided that $T > 0$, the condition $\Delta \geq 0$ implies that $\eta \geq 0$ and the matrices $[\beta_{(ij)}]_{4\times4}$, $[\kappa'_{(ij)}]_{2\times2}$ have all non-negative eigenvalues. This gives 7 inequalities among the 15 Class D transport coefficients, while the remaining 8 are completely arbitrary.

In summary, the most generic constitutive relations of a $(3+1)$-dimensional relativistic superfluid up to one-derivative order are given as: the energy-momentum tensor

$$
\begin{aligned}
T^{\mu\nu} &= (E+P)u^\mu u^\nu + Pg^{\mu\nu} + R_s\xi^\mu\xi^\nu \\
&+ u^\mu u^\nu \left[\sum_{i=1}^{2}\alpha_{E,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{E,i}\tilde{S}_{e,i} - \frac{1}{T}\mathrm{D}_\sigma(Tf_1\zeta^\sigma) + \epsilon^{\alpha\rho\sigma\tau}u_\alpha\mathrm{D}_\rho\left(Tg_1u_\sigma\zeta_\tau\right)\right] \\
&+ 2u^{(\mu}\zeta^{\nu)}\left[\sum_{i=1}^{2} f_iS_{4+i} - \mu\left(\sum_{i=1}^{2}\alpha_{R_s,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i}\right) + \frac{1}{2\hat{\mu}_s}\epsilon^{\alpha\rho\sigma\tau}\zeta_\alpha\mathrm{D}_\rho\left(Tg_1u_\sigma\zeta_\tau\right)\right] \\
&+ \zeta^\mu\zeta^\nu\left[\sum_{i=1}^{2}\alpha_{R_s,i}S_{e,i} + \sum_{i=1}^{2}\left(\tilde{\alpha}_{R_s,i} - \frac{g_i}{2\hat{\mu}_s}\right)\tilde{S}_{e,i} - \sum_{i=1}^{4}\beta_{2i}S_i\right] \\
&+ 2u^{(\mu}\left[\mu\sum_{i=1}^{2} f_iV_{e,i}^{\nu)} - \sum_{i=1}^{2} g_i\tilde{V}_{e,2+i}^{\nu)} - P_\zeta{}^{\nu)}{}_\alpha\epsilon^{\alpha\rho\sigma\tau}\mathrm{D}_\rho\left(Tg_1u_\sigma\zeta_\tau\right) + 2C_1T^3\omega^{\nu)}\right. \\
&\quad \left. + \mu^2C\left(3B^{\nu)} + 2\mu\omega^{\nu)}\right)\right] - 2\zeta^{(\mu}\left[\sum_{i=1}^{2} f_iV_{e,i}^{\nu)} + \sum_{i=1}^{2}\kappa_{1i}V_i^{\nu)} + \sum_{i=1}^{2}\tilde{\kappa}_{1i}\tilde{V}_i^{\nu)}\right] \\
&+ P_\zeta^{\mu\nu}\left[\sum_{i=1}^{2} f_iS_{e,i} - \sum_{i=1}^{4}\beta_{1i}S_i\right] - \eta\sigma_\zeta^{\mu\nu} - \tilde{\eta}\tilde{\sigma}_\zeta^{\mu\nu} + \mathcal{O}(\partial^2),
\end{aligned}
\tag{3.36}
$$

and the charge current

$$
\begin{aligned}
J^\mu &= Qu^\mu - R_s\xi^\mu \\
&+ u^\mu\left[\sum_{i=1}^{2}\alpha_{Q,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{Q,i}\tilde{S}_{e,i} - \frac{1}{T}\mathrm{D}_\nu(Tf_2\zeta^\nu) + \epsilon^{\alpha\nu\rho\sigma}u_\alpha\mathrm{D}_\nu(Tg_2u_\rho\zeta_\sigma)\right] \\
&- \zeta^\mu\left[\sum_{i=1}^{2}\alpha_{R_s,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i} + \sum_{i=1}^{4}\beta_{3i}S_i - \frac{1}{2\hat{\mu}_s}\epsilon^{\alpha\nu\rho\sigma}\zeta_\alpha\mathrm{D}_\nu(Tg_2u_\rho\zeta_\sigma)\right] \\
&+ \sum_{i=1}^{2} f_iV_{e,i}^\mu + \sum_{i=1}^{2} g_i\tilde{V}_{e,i}^\mu - \sum_{i=1}^{2}\kappa_{2i}V_i^\mu - \sum_{i=1}^{2}\tilde{\kappa}_{2i}\tilde{V}_i^\mu - P_\zeta{}^\mu{}_\alpha\epsilon^{\alpha\nu\rho\sigma}\mathrm{D}_\nu(Tg_2u_\rho\zeta_\sigma) \\
&+ 3\mu C^{(4)}\left(2B^\mu + \mu\omega^\mu\right) + \mathcal{O}(\partial^2).
\end{aligned}
\tag{3.37}
$$

The corrected Josephson equation is given by

$$
\begin{aligned}
u^\mu\xi_\mu &= \mu + \frac{1}{\beta_{44}}\mathrm{D}_\mu\left(R_s\xi^\mu\right) - \sum_{i=1}^{3}\frac{\beta_{4i}}{\beta_{44}}S_i \\
&+ \frac{1}{\beta_{44}}\mathrm{D}_\mu\left(\zeta^\mu\sum_{i=1}^{2}\alpha_{R_s,i}S_{e,i} + \zeta^\mu\sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i} - \sum_{i=1}^{2} f_iV_{e,i}^\mu - \sum_{i=1}^{2} g_i\tilde{V}_{e,i}^\mu\right) + \mathcal{O}(\partial^2),
\end{aligned}
\tag{3.38}
$$

which can be used to substitute for $S_4 = u^\mu\xi_\mu - \mu$ in the constitutive relations above. These constitutive relations satisfy the second law of thermodynamics with the free energy current

$$
\begin{aligned}
N^\mu = \frac{1}{T}Pu^\mu &+ \frac{1}{T}\sum_{i=1}^{2} f_i\left(u^\mu S_{e,i} - \zeta^\mu S_{4+i}\right) + \frac{1}{T}\sum_{i=1}^{2} g_i\left(u^\mu\tilde{S}_{e,i} + \tilde{V}_{2+i}^\mu\right) \\
&+ \frac{1}{T}3\mu^2C\,B^\mu + \frac{1}{T}\left(\mu^3C + 4T^3C_1\right)\omega^\mu.
\end{aligned}
\tag{3.39}
$$

We can also work out the associated entropy current explicitly

$$
\begin{aligned}
J^\mu_{\mathrm{S}} &= Su^\mu + g_1\frac{1}{T}\epsilon^{\mu\nu\rho\sigma}u_\nu\zeta_\rho\partial_\sigma T + g_2 T\epsilon^{\mu\nu\rho\sigma}u_\nu\xi_\rho\partial_\sigma\frac{\mu}{T} + 3C_1T^2\omega^\mu \\
&\quad + u^\mu\Bigg[\sum_{i=1}^{2}\alpha_{S,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{S,i}\tilde{S}_{e,i} - \frac{1}{T^2}\mathrm{D}_\sigma(Tf_1\zeta^\sigma) + \frac{\mu}{T^2}\mathrm{D}_\nu(Tf_2\zeta^\nu) \\
&\qquad + \frac{1}{T}\epsilon^{\alpha\nu\rho\sigma}u_\alpha\mathrm{D}_\nu\left(Tg_1u_\rho\zeta_\sigma\right) - \frac{\mu}{T}\epsilon^{\alpha\rho\sigma\tau}u_\alpha\mathrm{D}_\nu\left(Tg_2u_\rho\zeta_\sigma\right)\Bigg] \\
&\quad + \frac{1}{T}\zeta^\mu\Bigg[\sum_{i=1}^{4}\mu\beta_{3i}S_i + \frac{1}{2\hat{\mu}_s}\epsilon^{\alpha\rho\sigma\tau}\zeta_\alpha\mathrm{D}_\rho\left(Tg_1u_\sigma\zeta_\tau\right) - \frac{\mu}{2\hat{\mu}_s}\zeta_\alpha\epsilon^{\alpha\nu\rho\sigma}\mathrm{D}_\nu(Tg_2u_\rho\zeta_\sigma)\Bigg] \\
&\quad + \frac{\mu}{T}\sum_{i=1}^{2}\kappa_{2i}V^\mu_i + \frac{\mu}{T}\sum_{i=1}^{2}\tilde{\kappa}_{2i}\tilde{V}^\mu_i \\
&\qquad - \frac{1}{T}P_\zeta{}^\mu{}_\alpha\epsilon^{\alpha\nu\rho\sigma}\mathrm{D}_\nu\left(Tg_1u_\rho\zeta_\sigma\right) + \frac{\mu}{T}P_\zeta{}^\mu{}_\alpha\epsilon^{\alpha\nu\rho\sigma}\mathrm{D}_\nu(Tg_2u_\rho\zeta_\sigma).
\end{aligned} \tag{3.40}
$$

Including the ideal order pressure P, the superfluid constitutive relations up to the first derivative order are parametrised by 2 constants and 31 independent transport coefficients, 7 of which are restricted to be non-negative.

Similar to the ordinary fluids, we can also convert the superfluid constitutive relations to any desired hydrodynamic frame, for example the Landau frame. However, given the complexity of the constitutive relations as they are, we do not perform this exercise here.

3.2.4 Breaking of non-Abelian internal symmetries

In our discussion above, we focused on Abelian superfluids. However, in recent years (see e.g. [97]), non-Abelian superfluids have also started to attract some attention in the literature, in relation to the p-wave superfluidity observed in liquid ^{3}He [98, 99]. In this section, we explore how our Abelian results might be generalised to accommodate non-Abelian superfluids. The discussion presented here is directly from our work in [1].

Let us start with a quick recap of the non-Abelian spontaneous symmetry breaking. More details can be found in section 19 of [100]. Consider a microscopic theory which is invariant under spacetime Poincaré transformations and the global action of a semisimple Lie group G (with Lie algebra $\mathfrak{g}$). Let ψ be a field in the theory transforming under some unitary representation $\mathcal{D}(G)$ of G, i.e. under a $g \in G$ transformation $\psi \to \mathcal{D}(g)\psi$. The field ψ is said to spontaneously break the symmetry from G to its Lie subgroup $H \subset G$ (with Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$), if its ground state expectation value $\langle\psi\rangle$ is only invariant under H, i.e.

$$
\mathcal{D}(h)\left\langle\psi\right\rangle = \left\langle\psi\right\rangle \quad \text{if and only if} \quad h \in H. \tag{3.41}
$$

$\mathcal{D}(g)\left\langle\psi\right\rangle$ with $g \notin H$ represents the "other" ground states the system could have spontaneously chosen from. Around $\langle\psi\rangle$, the field ψ can be expressed as a group transformation of a reference field $\tilde{\psi}$, i.e. $\psi = \mathcal{D}(\gamma)\tilde{\psi}$, defined by

$$
\tilde{\psi}^\dagger\mathcal{D}(X)\left\langle\psi\right\rangle = \tilde{\psi}^\dagger\left\langle\psi\right\rangle, \qquad \forall\ X \in \mathfrak{g}. \tag{3.42}
$$

Roughly speaking, γ corresponds to the fluctuations of ψ which take us to the nearby ground states with no energy cost, while $\tilde{\psi}$ contains the genuine excitations of ψ.

Note that eq. (3.42) is invariant under $\tilde{\psi} \to \mathcal{D}(h)\tilde{\psi}$ with $h \in H$, and hence determines γ only up to a coset equivalence $\gamma \sim \gamma h$. Let us pick a representative from each coset $\gamma = \gamma(\varphi)$ parametrised by a field φ living in the Lie algebra quotient $\mathfrak{g}/\mathfrak{h}$. The field φ can be identified as the *Goldstone modes* of the broken symmetry. Under a $g \in G$ transformation, these modes transform according to

$$\gamma(\varphi) \to g\gamma(\varphi)h(\varphi,g)^{-1}, \quad \tilde{\psi} \to \mathcal{D}(h(\varphi,g))\tilde{\psi}, \tag{3.43}$$

for some $h(\varphi,g) \in H$, such that $\psi \to \mathcal{D}(g)\psi$ and eq. (3.42) remains invariant. From these transformation properties, it is clear that the theory cannot contain a mass term for φ, rendering it gapless.

Let us make contact with the Abelian case, where $G = \mathrm{U}(1)$ is broken down to $H = \{1\}$. The Goldstone mode φ is a scalar with $\gamma(\varphi) = \mathrm{e}^{-i\varphi}$. Under a $g = \mathrm{e}^{i\Lambda} \in \mathrm{U}(1)$ transformation, it transforms according to $\mathrm{e}^{-i\varphi} \to \mathrm{e}^{i\Lambda}\mathrm{e}^{-i\varphi}$ implying $\varphi \to \varphi - \Lambda$. This is what we used in our construction of Abelian superfluids.

Once we have our Goldstone modes ready, we need to define their covariant derivative, so as to define the associated superfluid velocity. However, there is a hitch; φ lives in the quotient $\mathfrak{g}/\mathfrak{h}$ and it is not very straightforward to define a covariant derivative on a quotient space. One such notion comes from the Maurer–Cartan form in differential geometry, however, we can use a much simpler setup for our purposes. Let us introduce a set of generators $\{\mathrm{t}_A\} = \{\mathrm{t}_i, \mathrm{t}_a\}$ of G such that the subset $\{\mathrm{t}_i\}$ generates H. We normalise these generators by choosing $\mathrm{t}_A \cdot \mathrm{t}_B = 2\mathrm{Tr}\,[\mathrm{t}_A\mathrm{t}_B] = \eta_{AB}$, where η_{AB} is a diagonal matrix with entries ± 1. We define the action of an element $g \in G$ on these generators as $\mathrm{t}_A \to \mathrm{Ad}_g(\mathrm{t}_A) = (\mathrm{Ad}_g)^B{}_A \mathrm{t}_B = g\mathrm{t}_A g^{-1}$. We can now use these to define a set of projectors

$$\mathrm{P}(\mathrm{t}_A) = \mathrm{P}^B{}_A \mathrm{t}_B = \left((\mathrm{Ad}_{\gamma^{-1}})^B_i (\mathrm{Ad}_\gamma)^i{}_A\right)\mathrm{t}_B, \qquad \overline{\mathrm{P}}(\mathrm{t}_A) = \overline{\mathrm{P}}^B{}_A \mathrm{t}_B = \left(\delta^B_A - \mathrm{P}^B{}_A\right)\mathrm{t}_B. \tag{3.44}$$

It can be checked that they covariantly project out the components of $X \in \mathfrak{g}$ along and against the residual symmetry respectively. Using these, we can package the non-covariant information in the derivatives of the Goldstone modes φ into a covariant object

$$\xi_\mu = \overline{\mathrm{P}}\left(i\partial_\mu\gamma(\varphi)\gamma(\varphi)^{-1} + A_\mu\right) \in \overline{\mathrm{P}}(\mathfrak{g}), \tag{3.45}$$

identified as the superfluid velocity. As an added benefit of the projectors we defined, we can revert back to the ordinary fluids at any point by setting $\overline{\mathrm{P}} = 0$, $\mathrm{P} = \mathrm{id}_{\mathfrak{g}}$ (identity in $\mathfrak{g}$).

From this point forward, the analysis is exactly parallel to our Abelian superfluid discussion. We introduce a set of equations of motion for φ

$$K \approx 0 \in \overline{\mathrm{P}}(\mathfrak{g}). \tag{3.46}$$

The adiabaticity equation for this system is given simply by

$$\mathrm{D}_\mu N^\mu - \mathrm{N}_{\mathrm{H}}^{\perp} = \frac{1}{2} T^{\mu\nu} \delta_{\mathscr{B}} g_{\mu\nu} + J^\mu \cdot \delta_{\mathscr{B}} A_\mu + K \cdot \delta_{\mathscr{B}} \varphi + \Delta, \qquad \Delta \geq 0. \tag{3.47}$$

Here we have defined $\delta_{\mathscr{B}}\varphi = \beta^\mu \xi_\mu - \overline{\mathrm{P}}(\mu)/T$. Closely following our Abelian superfluid calculation, at the ideal order we find the set of constitutive relations, free energy, and entropy currents for a non-Abelian superfluid to be

$$\begin{aligned}
T^{\mu\nu} &= (E+P)u^\mu u^\nu + P g^{\mu\nu} + \xi^\mu \cdot R_s \cdot \xi^\nu + \mathcal{O}(\partial), \\
J^\mu &= Q u^\mu - R_s \cdot \xi^\mu + \mathcal{O}(\partial), \\
K &= -\alpha \cdot \delta_{\mathscr{B}}\varphi + \mathrm{D}_\mu (R_s \xi^\mu) + [\xi_\mu, R_s \cdot \xi^\mu] + \mathcal{O}(\partial), \\
N^\mu &= \frac{1}{T} P u^\mu + \delta_{\mathscr{B}}\varphi \cdot R_s \cdot \xi^\mu + \mathcal{O}(\partial), \\
J^\mu_{\mathrm{S}} &= N^\mu - \frac{1}{T}\left(T^{\mu\nu} u_\nu + \mu \cdot J^\mu \right) = S u^\mu + \mathcal{O}(\partial).
\end{aligned} \tag{3.48}$$

Here $\alpha \in \overline{\mathrm{P}}(\mathfrak{g}) \times \overline{\mathrm{P}}(\mathfrak{g})$ is a positive semi-definite matrix. On the other hand, $R_s \in \overline{\mathrm{P}}(\mathfrak{g}) \times_{\mathrm{sym}} \overline{\mathrm{P}}(\mathfrak{g})$ is a symmetric matrix, which enters the thermodynamic relations

$$\begin{aligned}
\text{Gibbs-Duhem equation:} &\qquad \mathrm{d}P = S\mathrm{d}T + Q \cdot \mathrm{d}\mu + R_s^{AB} \mathrm{d}\mu^s_{AB}, \\
\text{Euler scaling relation:} &\qquad E + P = ST + Q \cdot \mu, \\
\text{First law of thermodynamics:} &\qquad \mathrm{d}E = T\mathrm{d}S + \mu \cdot \mathrm{d}Q - R_s^{AB} \mathrm{d}\mu^s_{AB},
\end{aligned} \tag{3.49}$$

where $\mu^s_{AB} = -\frac{1}{2} \xi^\mu_A \xi^\nu_B g_{\mu\nu}$. By setting $K \approx 0$, we can compute the non-Abelian Josephson equation

$$u^\mu \xi_\mu = \overline{\mathrm{P}}(\mu) + T\alpha' \cdot \Big(\mathrm{D}_\mu (R_s \xi^\mu) + [\xi_\mu, R_s \cdot \xi^\mu] \Big) + \mathcal{O}(\partial), \tag{3.50}$$

where $\alpha' \cdot \alpha = \overline{\mathrm{P}}$. Note, again, that this equation is only accurate at the zero derivative order. At the first derivative order, it can still admit further corrections.

We wrap up our discussion of non-Abelian superfluids here. We could explicitly write down the one-derivative corrections, but nothing out of the ordinary happens compared to the Abelian case.

3.3 | Relativistic fluid surfaces

For the last example of this chapter, let us consider the spontaneous breaking of a spacetime symmetry rather than an internal one to generate a gapless mode. In the two hydrodynamic systems that we discussed above, we assumed the fluid to be space-filling, i.e. we assumed that it extends infinitely in every direction. This is a reasonable assumption when describing physics deep in the bulk of a fluid, but starts to break down as we approach an interface or a boundary. The very existence of such a "surface" breaks the translational symmetry of the theory and considerably modifies the hydrodynamic spectrum. Let us call the Goldstone mode of this broken translational generator to be $f(x)$. It is just a scalar field with the transformation property $\delta_\chi f = \chi^\mu \partial_\mu f$. Just like the superfluids, we take f to be $\mathcal{O}(\partial^{-1})$

in the derivative expansion.

A priori, including f in our description breaks the translational symmetry at every spacetime point. This has been utilised elsewhere in the literature to model lattices and study phenomena like momentum relaxation; see e.g. [101]. The corresponding hydrodynamic constitutive relations can be derived straightforwardly following our superfluid analysis, by substituting $\partial_\mu f$ for ξ_μ. Here, however, we are interested in somehow localising the symmetry breaking into a thin spatial region, so as to form a surface. With this in mind, let us perform a coordinate transformation to bring the surface of interest to $f = 0$, and introduce a distribution functional $\theta(f)$ centred around it. We assume that $\theta(f)$ completely characterises the f dependence in the constitutive relations. We further assume that $\theta'(f)$ is positive and is only supported within a narrow region around $f = 0$. The width of this region characterises the thickness of the surface. In the limit that the surface is infinitesimally thin, $\theta(f)$ approaches the Heaviside step function, while $\theta'(f)$ approaches the Dirac delta function. We call $f > 0$ to be the "inside" of the fluid while $f < 0$ to be its "outside".

We take $\theta(f)$ to be $\mathcal{O}(\partial^0)$ in the hydrodynamic derivative expansion. Taking its spacetime derivative we find that $\partial_\mu\theta(f) = \theta'(f)\partial_\mu f$, which we can formally decompose into[1]

$$\begin{aligned} z_\mu &= -\frac{\partial_\mu\theta(f)}{\sqrt{g^{\mu\nu}\partial_\mu\theta(f)\partial_\nu\theta(f)}} = -\frac{\partial_\mu f}{\sqrt{g^{\mu\nu}\partial_\mu f\partial_\nu f}}, \\ \tilde{\delta}(f) &= -z^\mu\partial_\mu\theta(f) = \sqrt{g^{\mu\nu}\partial_\mu f\partial_\nu f}\,\theta'(f). \end{aligned} \tag{3.51}$$

Due to f being $\mathcal{O}(\partial^{-1})$, the inward pointing normal vector to the surface z_μ is clearly $\mathcal{O}(\partial^0)$, while the distribution $\tilde{\delta}(f)$ is $\mathcal{O}(\partial^1)$. Similarly, one can check that all the higher derivative tensor structures following from $\theta(f)$ can be represented in terms of the derivatives of z_μ along with a series of distributions

$$\tilde{\delta}^{(n)}(f) = (-)^{n+1} z^{\mu_1}\ldots z^{\mu_{n+1}}\mathrm{D}_{\mu_1}\ldots\mathrm{D}_{\mu_{n+1}}\theta(f), \tag{3.52}$$

with $\tilde{\delta}^{(0)}(f) = \tilde{\delta}(f)$. We should note that $\partial_\mu\theta(f) = \theta'(f)\partial_\mu f$ is only supported near the surface, so even though the value of z_μ does not depend on $\theta(f)$, it is only well defined when used in conjunction with $\tilde{\delta}^{(n)}(f)$. Another thing to note is that, as long as the surface remains thin enough, we can expand the products like $\tilde{\delta}^{(n)}(f)\tilde{\delta}^{(m)}(f)$ into an infinite series $\sum_r c_r\tilde{\delta}^{(r)}(f)$ for some coefficients c_r, so we do not need to consider such products independently.

Having set the stage, let us write down the adiabaticity equation for this setup

$$\mathrm{D}_\mu N^\mu - \mathrm{N}^{\perp}_{\mathrm{H}} = \frac{1}{2}T^{\mu\nu}\delta_{\mathscr{B}}g_{\mu\nu} + J^\mu\delta_{\mathscr{B}}A_\mu + \frac{Y\delta_{\mathscr{B}}f}{\sqrt{g^{\nu\rho}\partial_\nu f\partial_\rho f}} + \Delta, \qquad \Delta \geq 0. \tag{3.53}$$

[1]In the original references [6, 59], the symbol n_μ has been used for the normal vector to the surface. We replace it with z_μ to avoid confusion with the Newton-Cartan clock form.

Here we have chosen the equation of motion for f to be $Y/\sqrt{g^{\mu\nu}\partial_\mu f \partial_\nu f} \approx 0$, and defined

$$\frac{\delta_{\mathscr{B}} f}{\sqrt{g^{\nu\rho}\partial_\nu f \partial_\rho f}} = \frac{\beta^\mu \partial_\mu f}{\sqrt{g^{\nu\rho}\partial_\nu f \partial_\rho f}} = -\frac{1}{T} u^\mu z_\mu. \tag{3.54}$$

Unlike for superfluids, the adiabaticity equation (3.53) does not have any solution at zero derivatives. Sure, we could write down a term $\delta_{\mathscr{B}} f$ in Y, but we agreed that all the dependence on z_μ in the constitutive relations must be accompanied by $\tilde{\delta}^{(n)}(f)$. Moving on to the ideal order, the Class $\mathrm{H_S}$ constitutive relations are characterised by a free-energy density

$$\mathcal{N} = \theta(f) P_{\text{in}}(T,\mu) + \bar{\theta}(f) P_{\text{out}}(T,\mu), \tag{3.55}$$

where $\bar{\theta}(f) = 1 - \theta(f)$. Here P_{in} and P_{out} are the thermodynamic pressures inside and outside the surface respectively. The only other non-trivial class at ideal order is Class D with $Y \sim -\alpha\theta'(f)\delta_{\mathscr{B}} f$ for some non-negative coefficient α. Together, they imply a set of constitutive relations

$$\begin{aligned}
T^{\mu\nu} &= \theta(f) T^{\mu\nu}_{\text{in}} + \bar{\theta}(f) T^{\mu\nu}_{\text{out}} + \mathcal{O}(\partial), \\
J^{\mu} &= \theta(f) J^{\mu}_{\text{in}} + \bar{\theta}(f) J^{\mu}_{\text{out}} + \mathcal{O}(\partial), \\
Y &= \tilde{\delta}(f) \left(\frac{\alpha}{T} u^\mu z_\mu + (P_{\text{in}} - P_{\text{out}}) \right) + \mathcal{O}(\partial).
\end{aligned} \tag{3.56}$$

Here $T^{\mu\nu}_{\text{in/out}}$ and $J^{\mu}_{\text{in/out}}$ are the energy-momentum tensors and charge currents of the fluid inside/outside the surface respectively, worked out previously in eq. (3.4). We see that there is no dedicated contribution from the surface at the ideal order. On the other hand, we get our first version of the f equation of motion

$$u^\mu z_\mu \approx -\frac{T}{\alpha}\Delta P + \mathcal{O}(\partial), \qquad \text{where} \quad \Delta P = P_{\text{in}} - P_{\text{out}}. \tag{3.57}$$

It materialises the natural expectation that in time the surface moves towards the lower pressure. Since ΔP characterises the "change" in pressure across the surface, we take it to be $\mathcal{O}(\partial^1)$ in the derivative expansion. As a corollary of the f equation of motion therefore, $u^\mu z_\mu$ is not independent on-shell, but is determined in terms of the other one-derivative order scalars in the theory.

Things become more interesting when we include the one-derivative corrections, i.e. we find the constitutive relations that solve eq. (3.53) at two derivative order. In Class $\mathrm{H_S}$, the free-energy density gets improved to admit a surface tension $\gamma(T,\mu)$,

$$\mathcal{N} = \theta(f) P_{\text{in}}(T,\mu) + \bar{\theta}(f) P_{\text{out}}(T,\mu) - \tilde{\delta}(f)\gamma(T,\mu). \tag{3.58}$$

In addition, following our discussion in section 3.1.2, we also have two copies of Class D constitutive relations η, ζ, and σ, one on either sides of the surface, and just a single copy of Class A and $\mathrm{H_V}$ constants.[2] All the other classes are empty. This leads to a set of

[2] The (thermal) anomaly polynomial is characterised by a set of constants, which are not allowed to vary over the spacetime manifold. Therefore it is not clear if we can take these constants to be different inside and outside the surface.

constitutive relations

$$
\begin{aligned}
T^{\mu\nu} &= \theta(f)T^{\mu\nu}_{\text{in}} + \bar{\theta}(f)T^{\mu\nu}_{\text{out}} + \tilde{\delta}(f)\Big((E_{\text{sur}} - \gamma)\, u^\mu u^\nu - \gamma\, (g^{\mu\nu} - z^\mu z^\nu) \Big) + \mathcal{O}(\partial^2), \\
J^\mu &= \theta(f)J^\mu_{\text{in}} + \bar{\theta}(f)J^\mu_{\text{out}} + \tilde{\delta}(f)Q_{\text{sur}}u^\mu + \mathcal{O}(\partial^2), \\
Y &= \tilde{\delta}(f)\left(\frac{\alpha}{T}u^\mu z_\mu + (P_{\text{in}} - P_{\text{out}}) - \mathrm{D}_\mu\,(\gamma z^\mu) \right) + \mathcal{O}(\partial^2),
\end{aligned}
\tag{3.59}
$$

where we have defined

$$
-\mathrm{d}\gamma = S_{\text{sur}}\mathrm{d}T + Q_{\text{sur}}\mathrm{d}\mu, \qquad E_{\text{sur}} - \gamma = S_{\text{sur}}T + Q_{\text{sur}}\mu, \tag{3.60}
$$

while $T^{\mu\nu}_{\text{in/out}}$ and $J^\mu_{\text{in/out}}$ are defined according to eq. (3.11). We see that this time we do get some dedicated surface contributions to the energy-momentum tensor and charge current. More interestingly, the f equation of motion also admits a correction to take the form

$$
u^\mu z_\mu \approx -\frac{T}{\alpha}\,(\Delta P - \mathrm{D}_\mu\,(\gamma z^\mu)) + \mathcal{O}(\partial). \tag{3.61}
$$

With this, we see that the gradient of pressure still contributes to the time-evolution of the surface, but it can partially be balanced by a surface tension term. To express this equation in a more familiar form, consider that the fluid in question has a constant surface tension and focus on a static surface configuration with $u^\mu z_\mu \propto u^\mu \partial_\mu f = 0$. In this regime, the above equation reduces to

$$
\Delta P \approx \gamma \mathrm{D}_\mu z^\mu + \mathcal{O}(\partial). \tag{3.62}
$$

This can immediately be recognised as the Young-Laplace equation. So we see that eq. (3.61) is just the Young-Laplace equation generalised to the out-of-equilibrium fluid configurations. Higher derivative corrections to the hydrodynamic constitutive relations and the Young-Laplace equation can also be worked out in a similar manner.

This point of view to describe the surface dynamics of a fluid in terms of a scalar “shape” field f was introduced in [6, 59]. The results presented in [6] even combine the internal and spacetime symmetry breaking and present surface dynamics in a superfluid. Apart from the obvious generalisations, some new hybrid features arise on the surface of a superfluid owing to the emergence of the new ideal order structures $z_\mu \xi^\mu$ and $\epsilon^{\mu\nu\rho\sigma}u_\nu z_\rho \xi_\sigma$. We do not concern ourselves with these details here, instead, we just note that they can still be understood within the off-shell framework of hydrodynamics we have presented in this work. For more details, we refer the reader to the discussion in [6].

These are all the examples of relativistic hydrodynamics that we consider in this thesis. We comment on some other possible applications in chapter 6. In the next chapter, we revisit the rules of hydrodynamics when the underlying spacetime symmetry group is taken to be Galilean rather than Poincaré, which provides a framework for non-relativistic hydrodynamics.

4 | Galilean hydrodynamics

For most practical purposes, the world around us can be regarded as non-relativistic. So it is natural to formulate a non-relativistic version of hydrodynamics, which is more suited to our day-to-day applications ranging from hydraulics to biophysics. Non-relativistic physics emerges as an effective description of the more fundamental relativistic description of our universe, in a limit where the speed of light c is very large compared to the other characteristic speeds under consideration. For a typical relativistic system, however, taking such a limit (sending $c \to \infty$) turns out to be a notoriously non-trivial task to perform. Except in a few special cases, the non-relativistic limit is either not well defined or is not unique,[1] which forces us to resort to other methods. One such method is to study the non-relativistic theories independent of their relativistic parents, using as our guiding principle the Galilean symmetry emergent in the non-relativistic limit. This has, of course, been the conventional view of "non-relativistic" physics for centuries, long before Einstein came up with his theory of relativity in 1905 [103]. In this chapter, we take this approach and set up Galilean hydrodynamics starting from the Galilean symmetry algebra.

To approach Galilean symmetries in a tractable manner, we use the framework of null fluids that we formulated in a series of papers [2–5]. Null fluids are a one-higher dimensional embedding of Galilean fluids, which can be seen as anisotropic relativistic fluids in their own right. Given our handle on relativistic hydrodynamics from chapter 2, they provide an alternate and more natural framework to study Galilean fluids. In section 1.1.3 we motivated null fluids as an emergent structure starting from Galilean fluids. In this chapter, however, we introduce null hydrodynamics from a relativistic standpoint, drawing a comparison with relativistic hydrodynamics discussed in chapter 2 from time to time. Towards the end of the chapter, we return to how these results can be converted to the conventional Galilean hydrodynamics language. For a more detailed discussion on the top-down approach to covariantly organise Galilean symmetries into a higher dimensional structure, see [82].

4.1 | Galilean field theories and null backgrounds

4.1.1 From Poincaré to Galilean algebra

Let us start at the very beginning—symmetries. Galilean field theories are known (or defined) to transform covariantly under the action of the so-called *Galilean algebra*.[2] The

[1] For example, Maxwell's electromagnetism is known to have more than one non-relativistic limits [102].

[2] To be precise, this is actually a central extension of the Galilean algebra, sometimes known as the *Bargmann algebra*. Galilean algebra sits inside the Bargmann algebra as a special case with $M = 0$.

generators of this algebra are given as

$$\text{Continuity:}\quad M,\qquad \text{Time-translation:}\quad H\qquad \text{Translations:}\quad P_a,$$
$$\text{Galilean boosts:}\quad B_a,\qquad \text{Rotations:}\quad J_{ab}. \tag{4.1a}$$

The associated conserved quantities are the mass, energy, momenta, centre of mass velocity,[3] and angular momenta respectively. Here the indices $a, b, \ldots$ run over the $d-1$ flat spatial coordinates. For the familiar universe we live in, $d-1=3$, but it is useful to keep d arbitrary to allow for the study of non-relativistic systems which are effectively confined to a thin film ($d-1=2$) or a chain ($d-1=1$). On flat space, these generators have the following non-vanishing commutation relations

$$[B_a, H] = iP_a,\qquad [B_a, P_b] = i\delta_{ab}\, M,$$
$$[J_{ab}, P_c] = i\left(\delta_{ac}\, P_b - \delta_{bc}\, P_a\right),\qquad [J_{ab}, B_c] = i\left(\delta_{ac}\, B_b - \delta_{bc}\, B_a\right),$$
$$[J_{ab}, J_{cd}] = i\left(\delta_{ac}\, J_{bd} - \delta_{ad}\, J_{bc} - \delta_{bc}\, J_{ad} + \delta_{bd}\, J_{ac}\right), \tag{4.1b}$$

where δ_{ab} is the Kronecker delta. If this algebra is not familiar, it would help to consider it for a minute. Starting at the end, the rotation generators J_{ab} in the last line satisfy the standard SO(d−1) algebra. Furthermore, the commutators in the second line merely state the fact that the translation and boost generators P_a and B_a transform as SO(d−1) vectors. Since J_{ab} commutes with the remaining generators M and H, they transform as SO(d−1) scalars. We also see that when we commute a boost B_a with a time translation H, we end up generating a spatial translation. The characteristic feature of the Galilean algebra is the boost-translation commutator, which measures the mass of a state. This is the only place where M appears in the algebra and is therefore central.

Let us leave the Galilean algebra here for now and consider a $(d+1)$-dimensional Poincaré algebra with generators

$$\text{Spacetime translations:}\quad P_{\text{A}},\qquad \text{Lorentz transformations:}\quad J_{\text{AB}}. \tag{4.2a}$$

Here the indices $\text{A}, \text{B}, \ldots$ run over $d+1$ higher dimensional coordinates. They satisfy the usual commutation relations

$$[P_{\text{A}}, P_{\text{B}}] = 0,\qquad [J_{\text{AB}}, P_{\text{C}}] = i\left(\eta_{\text{AC}}\, P_{\text{B}} - \eta_{\text{BC}}\, P_{\text{A}}\right),$$
$$[J_{\text{AB}}, J_{\text{CD}}] = i\left(\eta_{\text{AC}}\, J_{\text{BD}} - \eta_{\text{AD}}\, J_{\text{BC}} - \eta_{\text{BC}}\, J_{\text{AD}} + \eta_{\text{BD}}\, J_{\text{AC}}\right), \tag{4.2b}$$

where η_{AB} is the pseudo-Riemannian flat Minkowski metric. The generators J_{AB} naturally span a Lorentz algebra SO$(d,1)$. The spacetime translation generators P_{A} are SO$(d,1)$ vectors and mutually commute. Let us choose a set of null coordinates $(x^{\text{A}}) = (x^-, x^+, x^a)$ such that $\eta_{++} = \eta_{--} = \eta_{+a} = \eta_{-a} = 0$, $\eta_{+-} = -1$, and $\eta_{ab} = \delta_{ab}$. In this coordinate system, consider a subset of the Poincaré generators, all of which commute with the null

[3] This conserved quantity is surprisingly much less talked about. The point particle version of this "conservation law" is that the centre of mass of a collection of particles moves at a constant velocity in any inertial reference frame.

momenta P_-. They are given by

$$M \equiv -P_-, \qquad H \equiv -P_+, \qquad P_a, \qquad B_a \equiv -M_{a-}, \qquad M_{ab}. \tag{4.3}$$

In fact, the only generators that do not make the cut are M_{A+}. We can check that these generators are closed under the commutation relations (4.2b), and exactly span the d-dimensional Galilean algebra (4.1). We, therefore, see that a d-dimensional Galilean algebra sits as a subalgebra in a $(d+1)$-dimensional Poincaré algebra, with one of the null momenta acting as the central mass generator. See [76] for an extensive review of this construction, and an extension to the Schrödinger algebra—Galilean analogue of a conformal algebra appearing naturally in non-relativistic holography [104]—arising from the null reduction of a $(d+1)$-dimensional relativistic conformal algebra.

This is rather convenient, as instead of starting from a d-dimensional relativistic theory and taking a $c \to \infty$ limit, or trying to write down a d-dimensional Galilean theory directly based on symmetries, we can start with a $(d+1)$-dimensional relativistic theory and reduce it over a light cone (introduce a null Killing vector) to get a Galilean theory. This is given the name *null reduction* [73–75] in the literature, also known as *light-cone reduction* or *discrete light-cone quantisation*. It has some obvious benefits over the other two approaches. We do not need to take a limit, so the prescription is perfectly well defined and unique. Also, we know how to deal with relativistic field theories reasonably well, which we can directly use to our advantage without having to deal with the technicalities of the much less understood Galilean field theories. The idea of null reduction has been used readily in the literature to reproduce known results and to get new insights into non-relativistic physics. Perhaps the most important of these results, at least in the current context, has been to reproduce Newton-Cartan geometries [77, 78] starting from a *Bargmann structure* (relativistic manifold carrying a covariantly constant null Killing vector) in one higher dimension; see e.g. [105–109]. Newton-Cartan geometries are a covariant representation of spacetime backgrounds which respect Galilean isometries. They are quite useful when describing Galilean physics as they treat space and time coordinates at the same footing, however, Galilean boost symmetry is not manifest in this formalism. As we go ahead, we provide a self-contained review of the aspects of it that we need.

4.1.2 Null backgrounds

As we did for relativistic hydrodynamics in section 2.1, let us start by setting up $(d+1)$-dimensional *null backgrounds* to which our null fluids are coupled. Roughly speaking, these are extensions of the Bargmann structures we mentioned above, to allow for non-trivial torsion and background gauge fields, in a way that they precisely capture the respective d-dimensional Galilean structure. Let the $(d+1)$-dimensional theory we are interested in be living on a spacetime manifold $\mathcal{M}$ with coordinates denoted by the indices $M, N, \ldots$. In this context, the indices $A, B, \ldots$ can be seen as coordinates on the frame bundle $F\mathcal{M}$ of $\mathcal{M}$. Given our Poincaré symmetries, Noether's theorem postulates the existence of a set of associated conserved currents

$$\text{Energy-momentum tensor:} \quad T^{M}{}_{A}\,, \qquad \text{Spin current:} \quad \Sigma^{MA}{}_{B}. \tag{4.4a}$$

For completeness, let us also introduce a Lie group G of internal symmetry which our theory of interest might enjoy. We call the associated Lie algebra $\mathfrak{g}$, which is endowed with a Lie bracket denoted by $[\circ\,,\circ]$ and a positive semi-definite inner product denoted by $\circ\cdot\circ$. The associated $\mathfrak{g}$ valued conserved current is taken to be

$$\text{Charge current:} \quad J^{M}. \tag{4.4b}$$

To probe these Noether currents, it is convenient to couple the theory to a set of relativistic background sources on $\mathcal{M}$, one corresponding to each current,

$$\begin{gathered}\text{Vielbein:} \quad e^{A}{}_{M}, \qquad \text{Spin connection:} \quad C^{A}{}_{MB}\,, \\ \text{Gauge connection:} \quad A_{M}.\end{gathered} \tag{4.5}$$

This is the exact same structure that we introduced in section 2.1 to study relativistic hydrodynamics. $e^{A}{}_{M}$ is a local frame field which furnishes an invertible map between the tangent bundle and the frame bundle. The spin connection $C^{A}{}_{MB}$ and the gauge connection A_M are one-form gauge fields valued in $\mathfrak{so}(d,1)$ and $\mathfrak{g}$ respectively. Along with an affine connection

$$\Gamma^{R}{}_{MS} \equiv e_{A}{}^{R}\left(\partial_M e^{A}{}_{S} + e^{B}{}_{S} C^{A}{}_{MB}\right), \tag{4.6}$$

they define a covariant derivative operator D_M on $\mathcal{M}$. Under an infinitesimal local Poincaré and G transformation parametrised by $\mathcal{X} = (\chi^M, \Lambda^{\Sigma}_{\chi}, \Lambda_{\chi})$, the variation of the background fields (4.5) is given by eq. (2.16) of our relativistic discussion.[4]

Recall that the Poincaré generator M_{A+} did not commute with the null momenta P_{-} and hence got left out when performing a null reduction to obtain the Galilean algebra. It follows that the respective Noether current $\Sigma^{MA}{}_{+}$ does not have any Galilean interpretation either. It holds to reason, therefore, that the associated background source $C^{A}{}_{M-}$ must also be switched off for consistency so that these "unphysical" relativistic currents are not probed. With a little bit of algebra, we can show that this implies

$$C^{A}{}_{M-} = e^{A}{}_{N}\left(\partial_M V^N + \Gamma^{N}{}_{MR} V^N\right) = e^{A}{}_{N} \mathrm{D}_M V^N = 0, \tag{4.7}$$

where $V^M = e_{-}{}^{M}$ is a null vector field. Furthermore, we should require that the background sources which do have a Galilean interpretation are left invariant under a local P_{-} transformation. Since P_{-} acts on the background by a Lie shift along V^M, it equivalently implies that V^M acts as an isometry on all the background fields. These are the defining features of the so-called *Bargmann structures*—spacetimes with a covariantly constant null isometry—which have been shown to be equivalent to the lower dimensional Newton-Cartan backgrounds, at least in the absence of an independent spin connection and gauge field.

[4] All the results from section 2.1 can be directly imported here by changing $d \to d+1$ and switching to the uppercase indices.

Before generalising this statement to backgrounds with arbitrary connections, let us do a quick counting exercise. After imposing $e^{A}{}_{M}V^{M} = \delta^{A}{}_{-}$, the vielbein $e^{A}{}_{M}$ has $d(d+1)$ independent components. d^2 of these make up a d-dimensional Galilean version of the vielbein, while the remaining d are identified with the components of a mass gauge field on a Newton-Cartan background. So this counting checks out. On the other hand, the spin connection $C^{A}{}_{MB}$, which is antisymmetric in $A \leftrightarrow B$ and satisfies $C^{A}{}_{M-} = 0$, has $d(d-1)/2$ independent components for each of the $d+1$ values that the spacetime index M can take. In contrast, the d-dimensional Galilean spin connections also have $d(d-1)/2$ components, but only for d distinct spacetime indices. The same goes for the gauge connection A_M as well, which has $d+1$ $\mathfrak{g}$-valued components as opposed to d expected in a d-dimensional Galilean theory. To have an exact mapping, therefore, we must somehow get rid of these extra components. Due to the presence of a preferred vector field V^M in the background, the only such components we can eliminate in a Poincaré invariant manner are $V^M C^{A}{}_{MB}$ and $V^M A_M$. They are not gauge-invariant, however, so we cannot merely set them to zero. Therefore, we introduce some arbitrary symmetry parameters $\Lambda^{\Sigma}_{V} \in \mathfrak{so}(d,1)$ and $\Lambda_V \in \mathfrak{g}$ on the background and require

$$V^M C^{A}{}_{MB} + \Lambda^{\Sigma}_{V}{}^{A}{}_{B} = V^M A_M + \Lambda_V = 0, \tag{4.8}$$

This eliminates the extra $d(d-1)/2$ components from the spin connection and a $\mathfrak{g}$-valued component from the gauge connection. We dub this requirement as the "compatibility of null isometry", and call the Bargmann structures that respect it to be *null backgrounds*. It is clear that these null backgrounds are in exact correspondence with the Newton-Cartan backgrounds, both in terms of the field content and symmetries, and as an added benefit have all the Galilean symmetries manifest in disguise of a Poincaré invariant structure.

For later convenience, let us recollect the definition of null backgrounds. We call a manifold $\mathcal{M}$ with relativistic background sources $(e^{A}{}_{M}, C^{A}{}_{MB}, A_M)$ to be a *null background* if it admits a preferred set of symmetry data $\mathcal{V} = (V^M, \Lambda^{\Sigma}_{V}, \Lambda_V)$ such that

1. Action of $\mathcal{V}$ is an isometry: $\delta_{\mathcal{V}} e^{A}{}_{M} = \delta_{\mathcal{V}} C^{A}{}_{MB} = \delta_{\mathcal{V}} A_M = 0$,
2. V^M is null: $V^M V_M = 0$,
3. V^M is covariantly constant: $\mathrm{D}_M V^N = 0$, and
4. $\mathcal{V}$ is compatible: $V^M A_M + \Lambda_V = 0$, $V^M C^{A}{}_{MB} + \Lambda^{\Sigma}_{V}{}^{A}{}_{B} = 0$.

Existence of the null isometry $\mathcal{V}$ leads to some nice features on null backgrounds. Firstly, contraction of the spacetime indices of the Riemann curvature tensor $R_{MN}{}^{A}{}_{B}$, field strength F_{MN}, and Cartan torsion tensor $\mathrm{T}^{A}{}_{MN}$ (defined in eq. (2.15)) with V^M vanishes

$$V^M \mathrm{T}^{A}{}_{MN} = V^M F_{MN} = V^M R_{MN}{}^{A}{}_{B} = 0. \tag{4.9a}$$

These follow trivially from the symmetry variations in eq. (2.16) upon substituting $\mathcal{V}$ for $\mathcal{X}$. Furthermore, contraction of the SO$(d,1)$ indices of $R_{MN}{}^{A}{}_{B}$ and $\mathrm{T}^{A}{}_{MN}$ with V^A is also determined to be

$$R_{MN}{}^{A}{}_{B} V_A = 0, \qquad \mathrm{T}^{A}{}_{MN} V_A = 2\partial_{[M} V_{N]} \equiv H_{MN}. \tag{4.9b}$$

These can also be easily derived using $\mathrm{D}_{[M}\mathrm{D}_{N]}V^R = \mathrm{D}_{[M}V_{N]} = 0$. The presence of these relations is quite natural; they ensure that the respective quantities have the correct number of independent components as we would expect in a d dimensional Galilean background. Finally, we note that if φ is an arbitrary tensor transforming in some well behaved representation of diff $\times\, \mathfrak{so}(d,1) \times \mathfrak{g}$, then

$$\delta_{\mathcal{V}}\varphi = V^M \mathrm{D}_M \varphi. \tag{4.10}$$

The easiest way to convince oneself of its validity is to see it work for some specific examples of φ. It essentially works because of the compatibility requirement (condition 4) imposed on the connections on a null background, and would not be generically true on a Bargmann structure. For most of this work, we are dealing with tensor structures which respect the null isometry $\mathcal{V}$, i.e. $\delta_{\mathcal{V}}\varphi = 0$. Through eq. (4.10), this is equivalent to the requirement that φ be covariantly constant along V^M.

4.1.3 Ward identities

Parallel to our relativistic discussion in section 2.1.3, we would like to draw some generic conclusions about the physical theories coupled to null backgrounds. Let the theory in question be described by some dynamical fields φ^I with associated equations of motion $\mathcal{E}_I \approx 0$, and some (effective) action $S[e^A{}_M, C^A{}_{MB}, A_M, \mathcal{V}; \varphi^I]$. To be consistent with our null backgrounds setup, we require the dynamical fields to respect the null isometry $\mathcal{V}$, i.e. $\delta_{\mathcal{V}}\varphi^I = V^M \mathrm{D}_M \varphi^I = 0$. This ensures that the fields φ^I only have a dependence on the d spacetime coordinates, which is what we expect for a Galilean theory. Consequently, all the observables in the theory, especially the conserved currents, are covariantly constant along V^M.

Under an infinitesimal symmetry transformation $\mathcal{X}$, demanding the variation of the action $\delta_{\mathcal{X}} S$ to be zero, modulo anomalies, leads to a set of identities similar to eq. (2.19),

$$\begin{aligned}
\underline{\mathrm{D}}_M T^M{}_A &= e_A{}^N \left(\mathrm{T}^B{}_{NM} T^M{}_B + R_{NM}{}^C{}_B \Sigma^{MB}{}_C + F_{NM} \cdot J^M \right) + \mathcal{O}^I_A \mathcal{E}_I, \\
\underline{\mathrm{D}}_M \Sigma^{MAB} &= T^{[BA]} + \Sigma^{\perp AB}_{\mathrm{H}} + \mathcal{O}^{IAB}\mathcal{E}_I, \\
\underline{\mathrm{D}}_M J^M &= \mathrm{J}^{\perp}_{\mathrm{H}} + \mathcal{O}^I \mathcal{E}_I,
\end{aligned} \tag{4.11}$$

where $\underline{\mathrm{D}}_M = \mathrm{D}_M + e_A{}^N \mathrm{T}^A{}_{MN}$. The Hall currents $\Sigma^{\perp A}_{\mathrm{H}}{}_B$ and $\mathrm{J}^{\perp}_{\mathrm{H}}$ appearing here characterise the possible anomalies; they are considered in detail in the next subsection. On-shell, when $\mathcal{E}_I \approx 0$, these identities reduce to the Noether conservation laws for various currents

$$\begin{aligned}
\underline{\mathrm{D}}_M T^M{}_A &\approx e_A{}^N \left(\mathrm{T}^B{}_{NM} T^M{}_B + R_{NM}{}^C{}_B \Sigma^{MB}{}_C + F_{NM} \cdot J^M \right), \\
\underline{\mathrm{D}}_M \Sigma^{MAB} &\approx T^{[BA]} + \Sigma^{\perp AB}_{\mathrm{H}}, \\
\underline{\mathrm{D}}_M J^M &\approx \mathrm{J}^{\perp}_{\mathrm{H}}.
\end{aligned} \tag{4.12}$$

These will be the starting point for our hydrodynamic discussion later.

Consider the following deformation of the Noether currents:

$$T^M{}_A \to T^M{}_A + V^M \theta_{1A}, \quad \Sigma^{MAB} \to \Sigma^{MAB} + V^M \theta_2^{[AB]} - \theta_3^{M[A} V^{B]}, \quad J^M \to J^M + V^M \theta_4, \quad (4.13)$$

where θ's are some arbitrary tensor structures. The energy-momentum and charge components in eqs. (4.11) and (4.12) remain invariant, while the spin equation changes by a term

$$\left(\theta_1^{[A} + \underline{\mathrm{D}}_M \theta_3^{M[A}\right) V^{B]}. \quad (4.14)$$

This term only affects the conservation laws for the spin current components $\Sigma^{MA}{}_+$ which, if we recall, do not show up in the lower dimensional Galilean spectrum. Therefore, as far as the Galilean currents are concerned, shifts given in eq. (4.13) are mere redundancies in the higher dimensional Noether currents. They are not really surprising either; they are merely illustrating the fact that a $(d+1)$-dimensional relativistic theory has many more observables compared to a d dimensional Galilean theory. There could be a family of relativistic theories that would give rise to the same Galilean theory upon null reduction. Ideally, we would like to be able to fix these redundancies once and for all by making some suitable choice, however there is no "$(d+1)$-dimensional covariant" choice we can make at this point. The procedure for projecting out the unphysical degrees of freedom is essentially the null reduction prescription which leads to Newton-Cartan geometries, as we discuss in section 4.2. When working in the null background formalism while discussing Galilean hydrodynamics, we let these redundancies be, to enjoy working in a manifestly covariant formalism of Galilean physics.

4.1.4 Galilean anomalies

In this subsection, we outline the anomaly inflow mechanism for Galilean field theories and use it to explicitly compute the Hall currents appearing in the conservation laws eq. (4.12). These results were originally discussed in our work in [5], which we directly import here. Similar to our relativistic treatment in section 2.1.4, we introduce a $(d+2)$-dimensional bulk manifold $\mathcal{B}$, on whose boundary our physical manifold $\mathcal{M}$ lives, except that $\mathcal{B}$ is also required to be a null background with compatible null isometry $\mathcal{V}$. The indices on $\mathcal{B}$ are denoted by a hat. At the boundary, we keep our physical theory with generating functional $W_{\mathcal{M}}$ while in the bulk we introduce a generating functional $W_{\mathcal{B}}$ given by

$$W_{\mathcal{B}} = \int_{\mathcal{B}} \boldsymbol{I}. \quad (4.15)$$

The full generating functional $W = W_{\mathcal{M}} + W_{\mathcal{B}}$ is required to be symmetry-invariant. For this to happen, the anomaly polynomial of the theory $\boldsymbol{\mathcal{P}} = \mathrm{d}\boldsymbol{I}$ should be closed, invariant under all the symmetries, and should not be expressible as the exterior derivative of a symmetry-invariant form.

Now comes the interesting part. On even dimensional $(d+1 = 2n)$ null backgrounds, the allowed anomaly polynomial would take the usual Chern-Simons structure of relativistic theories $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_{\mathrm{CS}}$, which is made out of the Chern classes of $\boldsymbol{F}$ and the Pontryagin classes of $\boldsymbol{R}$. Note however that neither of $\boldsymbol{F}$ or $\boldsymbol{R}$ has a component along V^M, hence $\boldsymbol{\mathcal{P}}_{\mathrm{CS}}$ is identically zero. This suggests that we cannot get anomalies in an even dimensional null theory, and hence odd dimensional Galilean field theories are anomaly-free. Since odd dimensional relativistic theories are anomaly free as well, this is in fact physically sensible. If we had not required the compatibility of our background connections with the null isometry, $V^M F_{MN}$ and $V^M R_{MN}{}^A{}_B$ would not be zero, and we would end up with anomalous odd-dimensional Galilean field theories. This was the case, for example, for the Galilean anomalies found in [110].

The problem is that if we now shift our attention to odd dimensional $(d+1 = 2n+1)$ null backgrounds, there is no Chern-Simons anomaly polynomial to start with. Consequently, our even dimensional Galilean theories are rendered anomaly-free as well. We presented a resolution to this problem in [3, 5], which we now outline. Let us introduce a vector field v^M such that $v^M v_M = 0$ and $v^M V_M = -1$, and define $\boldsymbol{v} = v_M \mathrm{d}x^M$. With the help of this, we can write down an odd-rank anomaly polynomial

$$\boldsymbol{\mathcal{P}} = \boldsymbol{v} \wedge \boldsymbol{\mathcal{P}}_{\mathrm{CS}}. \tag{4.16}$$

Although this expression makes explicit reference to v_M, one can show that it is invariant under its arbitrary redefinition $v_M \to v_M + \delta v_M$. This follows from the fact that the change δv_M does not have any component along V^M, due to the normalisation property $\delta(v_M V^M) = V^M \delta v_M = 0$. We can convince ourselves that after the introduction of v^M, there are no more terms which can be written in the anomaly polynomial. Returning to the generating functional integrand $\boldsymbol{I}$, it can only be defined in the so-called *transverse gauge* where $\Lambda^{\Sigma}_V = \Lambda_V = 0$ leading to $V^M C^A{}_{MB} = V^M A_M = 0$. In this gauge, $\boldsymbol{I} = -\boldsymbol{v} \wedge \boldsymbol{I}_{\mathrm{CS}}$. Taking a differential we obtain

$$\mathrm{d}\boldsymbol{I} = \boldsymbol{\mathcal{P}} - \mathrm{d}\boldsymbol{v} \wedge \boldsymbol{I}_{\mathrm{CS}} = \boldsymbol{\mathcal{P}}. \tag{4.17}$$

The second term in the middle expression is zero in the transverse gauge, as it does not have any component along V^M. Hence, we recover the proposed anomaly polynomial.

Having our anomaly polynomial in place, the rest of the story is an exact parallel of section 2.1.4. In particular, we can use eq. (2.25) to compute the Galilean *Hall currents* as

$$\star_{(d+2)} \boldsymbol{\Sigma}_{\mathrm{H}} = \boldsymbol{v} \wedge \frac{\partial \boldsymbol{\mathcal{P}}_{\mathrm{CS}}}{\partial \boldsymbol{R}}, \qquad \star_{(d+2)} \mathbf{J}_{\mathrm{H}} = \boldsymbol{v} \wedge \frac{\partial \boldsymbol{\mathcal{P}}_{\mathrm{CS}}}{\partial \boldsymbol{F}}. \tag{4.18}$$

We can also verify that $\boldsymbol{\Sigma}_{\mathrm{H}}{}^A{}_B V^B = 0$, as the $\mathrm{SO}(d,1)$ indices of $\boldsymbol{\Sigma}_{\mathrm{H}}{}^A{}_B$ come from $\boldsymbol{R}^A{}_B$, which have a zero contraction with V^B.

This finishes our discussion of $(d+1)$-dimensional null backgrounds. We have discussed their structure, symmetries, conservation laws, and equivalence to the lower dimensional Galilean backgrounds. In the next section, we illustrate their equivalence to Newton-Cartan backgrounds in d-dimensions.

4.2 Null reduction to Newton-Cartan backgrounds

Over the past decade, we have learned that the correct way to probe the conserved currents of a Galilean field theory is to couple it to a version of (torsional) Newton-Cartan backgrounds, in a way that keeps all the symmetries manifest. These backgrounds were first introduced in 1923 in a series of seminal papers by Élie Cartan [77, 78], to describe a covariant framework for Newtonian gravity. Since then, they have been continually developed and extended upon, particularly in the context of covariant Newtonian gravity and coupling to Galilean field theories; see e.g. [79–82, 111–121] and also [122–126] for some related topics. Alongside, it has also been known that Newton-Cartan backgrounds can be obtained via reducing a relativistic background along a null isometry [105–109]. In the following, we start from our null backgrounds, discussed in the previous section, and establish their equivalence to torsional Newton-Cartan backgrounds. This also provides us with a natural null reduction prescription for $(d+1)$-dimensional field theories coupled to null backgrounds down to d-dimensional Galilean field theories. We provide a self-contained review of the aspects of Newton-Cartan backgrounds that we require for our subsequent discussion of Galilean hydrodynamics. For more details, we recommend checking out the PhD thesis of Michael Geracie [127], where he discusses the most generic torsional Newton-Cartan geometries and their coupling to Galilean field theories.

Let us consider a foliation of the spacetime manifold as $\mathcal{M} = S^1 \times \mathcal{N}$, where we have compactified the V^M direction into an infinitesimal circle S^1. The resultant d-dimensional null hypersurface $\mathcal{N}$ represents a Galilean (Newton-Cartan) spacetime. As $\mathcal{N}$ is a null hypersurface, to perform such a foliation uniquely we also require an arbitrary null field v^M doubly normalised as $v^M v_M = 0$ and $v^M V_M = -1$. Since we are introducing this vector field by hand just to perform the reduction, the actual physical results are invariant under an arbitrary shift of v^M. This choice can be understood as providing a "Galilean frame of reference". When working with hydrodynamics, the null fluid velocity u^M preferentially provides such a frame of reference which we refer to as the "fluid frame". For now however, we stick to an abstract v^M for the sake of generality. Formally, we define null reduction as this choice of v^M and subsequent decomposition of the $(d+1)$-dimensional theory into an effective d-dimensional one.

Let us first choose a basis on $\mathcal{M}$ as $(x^M) = (x^\sim, x^\mu)$ and partially fix the $\mathrm{SO}(d,1)$ and G-transformation symmetries to set $\mathcal{V} = (\partial_\sim, 0, 0)$, i.e. $V^M = \delta^M{}_\sim$, $\Lambda_V^{\Sigma}{}^A{}_B = \Lambda_V = 0$. The condition $V^M \mathrm{D}_M = 0$ simply becomes $\mathrm{D}_\sim = 0$. At this point, we can already decompose the conservation laws as

$$\begin{aligned}
\underline{\mathrm{D}}_\mu T^\mu{}_A &\approx e_A{}^\nu \left(\mathrm{T}^B{}_{\nu\mu} T^\mu{}_B + R_{\nu\mu}{}^C{}_B \Sigma^{\mu B}{}_C + F_{\nu\mu} \cdot J^\mu \right), \\
\underline{\mathrm{D}}_\mu \Sigma^{\mu A}{}_B &\approx \frac{1}{2} \left(e_{B\mu} T^{\mu A} - e^A{}_\mu T^\mu{}_B \right) + \Sigma_{\mathrm{H}}^{\perp A}{}_B, \\
\underline{\mathrm{D}}_\mu J^\mu &\approx \mathrm{J}_{\mathrm{H}}^{\perp},
\end{aligned} \tag{4.19}$$

up to the ambiguities given in eq. (4.13). Here the background structure can be understood as a d-dimensional manifold $\mathcal{N}$ (with indices $\mu, \nu, \ldots$) with an "extended" $(d+1)$-dimensional frame bundle (with indices $\mathrm{A}, \mathrm{B}, \ldots$). Note that we have not introduced the field v^{M} yet, so the description is still fully covariant. As we discussed in an appendix of [5], the fact that the Galilean physics can be packaged into this "extended space" structure was first realised, for theories without anomalies or spin currents, by [82], where the authors arrived at this bottom-up starting from the generic d-dimensional Newton-Cartan backgrounds. See [127] for a review. The novelty in the null-background formalism is that the symmetries are nicely arranged in terms of a Poincaré invariant structure.

4.2.1 Newton-Cartan backgrounds

Let us now introduce the Galilean frame velocity field v^{M}. We pick a basis on the frame bundle $(x^{\mathrm{A}}) = (x^-, x^+, x^a)$ such that $V^{\mathrm{A}} = \delta^{\mathrm{A}}{}_-$ and $v^{\mathrm{A}} = \delta^{\mathrm{A}}{}_+$. To this end, we can choose a specific representation of the Minkowskian metric η_{AB} and decompose the background fields as

$$\eta_{\mathrm{AB}} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \delta_{ab} \end{pmatrix}, \qquad e^{\mathrm{A}}{}_{\mathrm{M}} = \begin{pmatrix} 1 & -b_\mu \\ 0 & n_\mu \\ 0 & e^a{}_\mu \end{pmatrix},$$
$$C^{\mathrm{A}}{}_{\sim\mathrm{B}} = 0, \qquad C^{\mathrm{A}}{}_{\mu\mathrm{B}} = \begin{pmatrix} 0 & 0 & c_{\mu b} \\ 0 & 0 & 0 \\ 0 & c_\mu{}^a & C^a{}_{\mu b} \end{pmatrix}, \qquad A_{\mathrm{M}} = \begin{pmatrix} 0 \\ A_\mu \end{pmatrix}, \tag{4.20}$$

such that $n_\mu v^\mu = 1$ and $e^a{}_\mu v^\mu = 0$. The tensor structures appearing in this decomposition define a Newton-Cartan background

$$\text{Mass gauge field:} \quad b_\mu, \qquad \text{Clock form:} \quad n_\mu \qquad \text{Spatial vielbein:} \quad e^a{}_\mu,$$
$$\text{Temporal spin connection:} \quad c_\mu{}^a, \qquad \text{Spatial spin connection:} \quad C^a{}_{\mu b},$$
$$\text{Gauge connection:} \quad A_\mu. \tag{4.21}$$

They serve as the background sources associated with the Galilean symmetry generators given in eq. (4.1). n_μ serves as the "time-vielbein" on Galilean backgrounds, while $e^a{}_\mu$ is the spatial vielbein. b_μ is an external U(1) gauge field associated with the conservation of mass, while the connections $A_\mu \in \mathfrak{g}$ and $C^a{}_{\mu b} \in \mathfrak{so}(d{-}1)$ are associated with G-transformations and spatial rotations respectively. Finally, $c_\mu{}^a$ couples to the Galilean boosts and can be interpreted as the acceleration of the possibly non-inertial lab frame. Using the vielbein $e^a{}_\mu$, we can define an inverse vielbein $e_a{}^\mu$ via $e^a{}_\mu e_b{}^\mu = \delta^a{}_b$ and $e_a{}^\mu n_\mu = 0$. Together they furnish a resolution of identity

$$e_a{}^\mu e^a{}_\nu + v^\mu n_\nu = \delta^\mu{}_\nu, \tag{4.22}$$

and can be used to decompose the higher dimensional inverse vielbein

$$e_{\mathrm{A}}{}^{\mathrm{M}} = \begin{pmatrix} 1 & 0 \\ b_\nu v^\nu & v^\mu \\ b_\nu e_a{}^\nu & e_a{}^\mu \end{pmatrix}. \tag{4.23}$$

We can also define a degenerate spatial projection operator (also called spatial metric) via $h_{\mu\nu} = \delta_{ab} e^a{}_\mu e^b{}_\nu$, $h^{\mu\nu} = \delta^{ab} e_a{}^\mu e_b{}^\nu$, and $h^\mu{}_\nu = h^{\mu\sigma} h_{\sigma\nu} = e_a{}^\mu e^a{}_\nu$. They satisfy $h^{\mu\nu} n_\mu = h_{\mu\nu} v^\nu = 0$. Since there is no non-degenerate metric on $\mathcal{N}$, raising/lowering of $\mu, \nu, \ldots$ indices is not permitted. However $a, b, \ldots$ indices can be raised/lowered using the Kronecker delta δ_{ab}. On the other hand, the vielbeins $e^a{}_\mu$ and $e_a{}^\mu$ can be used to project the tensors on $\mathcal{N}$ to tensors on $\mathbb{R}^{(d-1)}$, and in turn lift the tensors on $\mathbb{R}^{(d-1)}$ to "spatial tensors" on $\mathcal{N}$ (whose n_μ or v^μ contraction is zero).

Decomposing the null background affine connection, we find that the only non-trivial components are

$$\Gamma^{R}{}_{\mu N} = \begin{pmatrix} 0 & c_\mu{}^a e_{a\nu} - \tilde{\mathrm{D}}_\mu b_\nu \\ 0 & \Gamma^\lambda{}_{\mu\nu} = v^\lambda \partial_\mu n_\nu + e_a{}^\lambda \left(e^b{}_\nu C^a{}_{\mu b} + n_\nu c_\mu{}^a + \partial_\mu e^a{}_\nu \right) \end{pmatrix}. \tag{4.24}$$

Here we have identified $\Gamma^\lambda{}_{\mu\nu}$ as the Newton-Cartan affine connection, which along with $C^a{}_{\mu b}$ and A_μ, defines a covariant derivative operator $\tilde{\mathrm{D}}_\mu$. We can check that $\Gamma^\lambda{}_{\mu\nu}$ is the unique affine connection that satisfies

$$\tilde{\mathrm{D}}_\mu n_\nu = 0, \qquad \tilde{\mathrm{D}}_\mu e^a{}_\nu + n_\nu c_\mu{}^a = 0 \iff e^a{}_\nu \tilde{\mathrm{D}}_\mu v^\nu = c_\mu{}^a, \qquad \tilde{\mathrm{D}}_\mu e_a{}^\nu = 0, \tag{4.25}$$

which are the Galilean equivalent of the metric compatibility condition. From here, we see that $c_\mu{}^a$ is indeed the background frame acceleration. For later use, let us define the *globally inertial Galilean frames* as those for which $c_\mu{}^a$ identically vanishes. However, for a generically curved Newton-Cartan background, there is no guarantee that such a frame would exist.

Under the null background symmetry transformations derived from eq. (2.16), the Newton-Cartan background fields (4.21) transform as

$$\begin{aligned}
\delta_{\mathcal{X}} b_\mu &= \pounds_\chi b_\mu + \partial_\mu \Lambda^{\mathrm{m}}_\chi + \Lambda^\tau_{\chi a}\, e^a{}_\mu \\
\delta_{\mathcal{X}} n_\mu &= \pounds_\chi n_\mu \\
\delta_{\mathcal{X}} e^a{}_\mu &= \pounds_\chi e^a{}_\mu - \Lambda^{\sigma\,a}_\chi{}_b\, e^b{}_\mu - \Lambda^{\tau\,a}_\chi\, n_\mu, \\
\delta_{\mathcal{X}} c_\mu{}^a &= \pounds_\chi c_\mu{}^a - \Lambda^{\sigma\,a}_\chi{}_b\, c_\mu{}^b + \partial_\mu \Lambda^{\tau\,a}_\chi + C^a{}_{\mu b} \Lambda^{\tau\,b}_\chi, \\
\delta_{\mathcal{X}} C^a{}_{\mu b} &= \pounds_\chi C^a{}_{\mu b} + \partial_\mu \Lambda^{\sigma\,a}_\chi{}_b + [C_\mu, \Lambda^\sigma_\chi]^a{}_b, \\
\delta_{\mathcal{X}} A_\mu &= \pounds_\chi A_\mu + \partial_\mu \Lambda_\chi + [A_\mu, \Lambda_\chi].
\end{aligned} \tag{4.26}$$

provided that we identify the Galilean symmetry parameters as

$$\mathcal{X} = \left(\ \chi^\mu\ ,\ \ \Lambda^{\mathrm{m}}_\chi = -\chi^\sim\ ,\ \ \Lambda^{\tau\,a}_\chi = \Lambda^{\Sigma\,a}_\chi{}_+\ ,\ \ \Lambda^{\sigma\,a}_\chi{}_b = \Lambda^{\Sigma\,a}_\chi{}_b\ ,\ \ \Lambda_\chi\ \right). \tag{4.27}$$

We can also work out the transformation properties of v^μ and $e_a{}^\mu$ as

$$\delta_{\mathcal{X}} v^\mu = \pounds_\chi v^\mu + e_a{}^\mu \Lambda^{\tau\,a}_\chi, \qquad \delta_{\mathcal{X}} e_a{}^\mu = \pounds_\chi e_a{}^\mu + \Lambda^{\sigma\,b}_\chi{}_a e_b{}^\mu. \tag{4.28}$$

Note that the diffeomorphisms along V^M take the form of mass gauge transformations on the Newton-Cartan backgrounds, justifying the name "mass gauge field" for b_μ. On the other hand, $\Lambda_\chi \in \mathfrak{g}$ acts as a G-gauge transformation on the respective gauge field A_μ. The action of χ^μ is a spacetime diffeomorphism, i.e. it acts on all the background fields by a Lie drag, as we would expect. The action of $\Lambda^\sigma_\chi \in \mathfrak{so}(d-1)$ is also simple; it transforms $C^a{}_{\mu b}$ as a gauge field and rotates the SO$(d-1)$ vectors $e^a{}_\mu$, $c_\mu{}^a$, and $e_a{}^\mu$ appropriately. Finally, we have the Galilean boost transformations parametrised by $\Lambda^\tau_\chi{}^a$, which mix up the background fields non-trivially. They are also known as Milne boosts in the literature.

We can null reduce the null background torsion, curvature, and field strength into the Newton-Cartan language. The only surviving components are

$$F_{\mu\nu}, \qquad \mathrm{T}^A{}_{\mu\nu} = \begin{pmatrix} -B_{\mu\nu} \\ -H_{\mu\nu} \\ \mathrm{T}^a{}_{\mu\nu} \end{pmatrix}, \qquad R_{\mu\nu}{}^A{}_B = \begin{pmatrix} 0 & 0 & -R_{\mu\nu b} \\ 0 & 0 & 0 \\ 0 & -R_{\mu\nu}{}^a & R_{\mu\nu}{}^a{}_b \end{pmatrix}. \tag{4.29}$$

Here we have defined

$$\begin{gathered} \text{Mass torsion:} \quad B_{\mu\nu} = 2\tilde{\mathrm{D}}_{[\mu} b_{\nu]} - \Omega_{\mu\nu}, \qquad \text{Spacetime torsion:} \quad \mathrm{T}^\lambda{}_{\mu\nu} = 2\Gamma^\lambda{}_{[\mu\nu]}, \\ \text{Spacetime curvature:} \quad R_{\mu\nu}{}^\lambda{}_\rho = 2\left(\partial_{[\mu}\Gamma^\lambda{}_{\nu]\rho} + \Gamma^\lambda{}_{[\mu|\sigma}\Gamma^\sigma{}_{\nu]\rho}\right), \\ \text{Field strength:} \quad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu], \end{gathered} \tag{4.30}$$

where $\Omega_{\mu\nu} = 2e_{a[\nu}c_{\mu]}{}^a$ is the (boost-non-invariant) Galilean frame vorticity, and further

$$\begin{gathered} H_{\mu\nu} = -n_\lambda \mathrm{T}^\lambda{}_{\mu\nu} = -2\partial_{[\mu}n_{\nu]}, \qquad \mathrm{T}^a{}_{\mu\nu} = e^a{}_\lambda \mathrm{T}^\lambda{}_{\mu\nu} = 2\partial_{[\mu}e^a{}_{\nu]} + 2c_{[\mu}{}^a n_{\nu]} + 2C^a{}_{[\mu b}e^b{}_{\nu]}, \\ R_{\mu\nu}{}^a = -R_{\mu\nu}{}^\lambda{}_\rho e^a{}_\lambda v^\rho = -2\left(\partial_{[\mu}c_{\nu]}{}^a + C^a{}_{[\mu b}c_{\nu]}{}^b\right), \\ R_{\mu\nu}{}^a{}_b = R_{\mu\nu}{}^\lambda{}_\rho e^a{}_\lambda e_b{}^\rho = 2\left(\partial_{[\mu}C^a{}_{\nu]b} + C^a{}_{[\mu c}C^c{}_{\nu]b}\right). \end{gathered} \tag{4.31}$$

Using these definitions, we can rewrite the Newton-Cartan affine connection given in eq. (4.24), into a more standard form found in the literature

$$\begin{aligned} \Gamma^\lambda{}_{\mu\nu} = v^\lambda \partial_\mu n_\nu &+ \frac{1}{2}h^{\lambda\sigma}\left(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right) \\ &+ n_{(\mu}\Omega_{\nu)\sigma}h^{\lambda\sigma} + \frac{1}{2}\left(e_a{}^\lambda \mathrm{T}^a{}_{\mu\nu} - 2e_{a(\nu}\mathrm{T}^a{}_{\mu)\sigma}h^{\lambda\sigma}\right). \end{aligned} \tag{4.32}$$

As far as we are aware, this was first written down in [82].[5]

Finally, using the $(d+1)$-dimensional volume form $\epsilon^{MN\cdots}$, we can define the raised and

[5] Let us say that our background contains a scalar field χ which transforms as $\delta_{\mathcal{X}}\chi = -\Lambda^{\mathrm{m}}_\chi$. We can use it to define boost-invariant torsion tensors $B_{\mu\nu} = v^\lambda M_\lambda H_{\mu\nu}$ and $\mathrm{T}^\lambda{}_{\mu\nu} = \left(h^{\lambda\sigma}M_\sigma - v^\lambda\right)H_{\mu\nu}$ where $M_\mu = b_\mu + \partial_\mu\chi$. For these special class of Newton-Cartan backgrounds, the affine connection can be expressed as

$$\Gamma^\lambda{}_{\mu\nu} = \left(v^\lambda - h^{\lambda\sigma}M_\sigma\right)\partial_\mu n_\nu + \frac{1}{2}h^{\lambda\sigma}\left(\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu}\right), \tag{4.33}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} + 2n_{(\mu}M_{\nu)}$. These backgrounds, termed as torsional Newton-Cartan geometries (TNC) in the literature [108, 128], appear naturally in the holography of Lifshitz spacetimes. A further restriction on $H_{\mu\nu}$ setting $h^{\mu\rho}h^{\nu\sigma}H_{\rho\sigma} = 0$ leads to the so called twistless torsional Newton-Cartan geometries (TTNC). If we were to switch off $H_{\mu\nu}$ entirely, we would recover the torsionless Newton-Cartan geometries.

lowered Newton-Cartan volume elements

$$\varepsilon^{\mu\nu\cdots} = v_M \epsilon^{M\mu\nu\cdots} = -\epsilon^{\sim\mu\nu\cdots}, \qquad \varepsilon_{\mu\nu\ldots} = V^M \epsilon_{M\mu\nu\ldots} = \epsilon_{\sim\mu\nu\ldots}. \tag{4.34}$$

Even though one of them involves an explicit mention of v^M, they are actually independent under an arbitrary frame redefinition. Note that since there is no raising/lowering operator on Newton-Cartan backgrounds, these two volume forms are independent. We can also define an associated Hodge duality operator $\star$, that uses $\varepsilon^{\mu\nu\cdots}$ to map m-rank differential forms to $(d-m)$-rank contravariant forms, while conversely $\varepsilon_{\mu\nu\ldots}$ to map m-rank contravariant forms to $(d-m)$-rank differential forms. It can be checked that $\star^2 = (-)^{m(d-m)}$ when acting on an m-rank differential or contravariant form.

4.2.2 Noether currents and conservation equations

Let us now move on to the Noether currents $T^M{}_A$, $\Sigma^{MA}{}_B$, and J^M. We start by recalling that there is a redundancy in the definition of these currents given by eq. (4.13), which allows us to eliminate some components. In the following decomposition, we have denoted these unphysical components by "$\times$", leading to

$$T^M{}_A = \begin{pmatrix} \times & \times & \times \\ -\rho^\mu & -\epsilon^\mu & p^\mu{}_a \end{pmatrix}, \qquad J^M = \begin{pmatrix} \times \\ j^\mu \end{pmatrix},$$
$$\Sigma^{\sim A}{}_B = \times, \qquad \Sigma^{\mu A}{}_B = \begin{pmatrix} \times & 0 & \times \\ 0 & \times & -\tau^\mu{}_b \\ -\tau^{\mu a} & \times & \sigma^{\mu a}{}_b \end{pmatrix}. \tag{4.35}$$

From here we can identify the Noether currents associated with the Galilean symmetry generators given in (4.1)

$$\text{Mass current:}\quad \rho^\mu, \qquad \text{Energy current:}\quad \epsilon^\mu \qquad \text{Momentum current:}\quad p^\mu{}_a,$$
$$\text{Temporal spin current:}\quad \tau^\mu{}_a, \qquad \text{Spatial spin current:}\quad \sigma^{\mu a}{}_b,$$
$$\text{Charge current:}\quad j^\mu. \tag{4.36}$$

With these identifications, the physical components of the Noether conservation laws (4.12) can be converted into the respective conservation laws of a Galilean theory with spin current

$$\begin{aligned}
\text{Mass conservation (continuity):}\quad & \underline{\tilde{\mathrm{D}}}_\mu \rho^\mu \approx 0, \\
\text{Energy conservation (time translation):}\quad & \underline{\tilde{\mathrm{D}}}_\mu \epsilon^\mu \approx -v^\mu f_\mu - p^\mu{}_a c_\mu{}^a, \\
\text{Momentum conservation (spatial translations):}\quad & \underline{\tilde{\mathrm{D}}}_\mu p^\mu{}_a \approx e_a{}^\mu f_\mu - \rho^\mu c_{\mu a}, \\
\text{Temporal spin conservation (Galilean boosts):}\quad & \underline{\tilde{\mathrm{D}}}_\mu \tau^\mu{}_a \approx \frac{1}{2}\left(n_\mu p^\mu{}_a - e_{a\mu}\rho^\mu\right), \\
\text{Spatial spin conservation (rotations):}\quad & \underline{\tilde{\mathrm{D}}}_\mu \sigma^{\mu ab} \approx p^{\mu[a} e^{b]}{}_\mu - 2\tau^{\mu[a} c_\mu{}^{b]} + \boldsymbol{\sigma}_{\mathrm{H}}^{\perp ab}, \\
\text{Charge conservation (G-transformations):}\quad & \underline{\tilde{\mathrm{D}}}_\mu j^\mu \approx \mathrm{j}_{\mathrm{H}}^{\perp}.
\end{aligned} \tag{4.37}$$

Here $\underline{\tilde{\mathrm{D}}}_\mu = \tilde{\mathrm{D}}_\mu + v^\nu H_{\nu\mu} - e_a{}^\nu \mathrm{T}^a{}_{\nu\mu}$, while f_μ denotes the external Lorentz force due to the presence of background fields

$$f_\mu = H_{\mu\nu}\epsilon^\nu + B_{\mu\nu}\rho^\nu + \mathrm{T}^a{}_{\mu\nu} p^\nu{}_a + R_{\nu\mu}{}^a \tau^\nu{}_a + R_{\mu\nu}{}^a{}_b \sigma^{\nu b}{}_a + F_{\mu\nu} \cdot j^\mu, \tag{4.38}$$

which act as a Galilean energy and momentum source. The terms coupling to $c_{\mu a} = e_{a\nu} \underline{\tilde{\mathrm{D}}}_\mu v^\nu$ in eq. (4.37) are due to the chosen Galilean frame v^μ not being globally inertial. Finally, we have the Galilean Hall currents $\boldsymbol{\sigma}_{\mathrm{H}}^{\perp ab} = \Sigma_{\mathrm{H}}^{\perp ab}$ and $\mathrm{j}_{\mathrm{H}}^{\perp} = \mathrm{J}_{\mathrm{H}}^{\perp}$, which can be explicitly computed via the reduction of eq. (4.18) leading to

$$\mathrm{j}_{\mathrm{H}}^{\perp} = -\star\left(\frac{\partial \boldsymbol{\mathcal{P}}_{\mathrm{CS}}}{\partial \boldsymbol{F}}\right), \qquad \boldsymbol{\sigma}_{\mathrm{H}}^{\perp a}{}_b = -\star\left(\frac{\partial \boldsymbol{\mathcal{P}}_{\mathrm{CS}}}{\partial \boldsymbol{R}^b{}_a}\right). \tag{4.39}$$

Eq. (4.37) are the complete set of conservation equations for a Galilean invariant field theory coupled to a curved spacetime background. In particular, note the temporal spin conservation equation. It implies that in the absence of a temporal spin current, the mass flux $\rho_a = e_{a\mu}\rho^\mu$ must equal the momentum density $p_a = n_\mu p^\mu{}_a$ in a Galilean theory, which is commonplace in non-relativistic physics.

To get some intuition, it is helpful to see these conservation laws on a flat Galilean background, where all the technicalities of Newton-Cartan backgrounds drop out. To do this, let us choose a basis $(x^\mu) = (t, x^i)$ such that $n_\mu = \partial_\mu t$. We choose to work in a frame given by $v^\mu = \delta^\mu{}_t$, and set $C^a{}_{\mu b} = c_\mu{}^a = b_\mu = 0$, and $e^a{}_\mu = \delta^a{}_\mu$. Let us also choose the background gauge field A_μ to be Abelian and restrict to $d = 4$ for familiarity. In this simplified case, the conservation laws are given as

$$\begin{gathered}
\partial_t \rho^t + \partial_i \rho^i = 0, \qquad \partial_t \epsilon^t + \partial_i \epsilon^i = E_i\, j^i, \qquad \partial_t p^t{}_k + \partial_i p^i{}_k = q\, E_k + \epsilon_{kij} j^i B^j, \\
\partial_t \tau^t{}_k + \partial_i \tau^i{}_k = \frac{1}{2}\left(\rho_k - p^t{}_k\right), \qquad \partial_t \sigma^{tjk} + \partial_i \sigma^{ijk} = p^{[kj]}, \\
\partial_t q^t + \partial_i j^i = 0.
\end{gathered} \tag{4.40}$$

Here we have defined the electric field $E_i = F_{it}$ and the magnetic field $B_i = \frac{1}{2}\epsilon_{ijk}F^{jk}$. The first and last equations are obviously the continuity equations for mass and charge respectively. The second equation is the continuity equation for energy, telling us that the rate of change of energy is proportional to the work done by the electric fields. The third equation is the Navier-Stokes equation for balancing force; note that the term on the right is merely the Lorentz force due to the electromagnetic fields. Finally, the two equations in the middle line are the conservation equations for the spin currents. For theories with no spin current, they merely tell us that the mass flux is same as the momentum density, and the stress tensor is symmetric.

4.2.3 Reference frame transformations

Recall that we had introduced an arbitrary vector field v^M to facilitate the null reduction. Since the relativistic parent theory does not depend on this choice of v^M, after all is said and done, we will be left with an additional v^M redefinition freedom in our Galilean theory. This is equivalent to the choice of Galilean reference frames. Let us see how various Galilean quantities defined above transform under this frame transformation: $v^M \to v^M + \psi^M$. Firstly, the normalisation conditions of v^M imply that

$$V^M \psi_M = 0, \qquad v^M \psi_M = -\frac{1}{2}\psi^M \psi_M. \tag{4.41}$$

While we do this transformation, we would still like the tangent space vector v^A to be equal to $\delta^A{}_+$. To achieve this, we need to follow this transformation by a tangent space rotation with matrix

$$O^A{}_B = \delta^A{}_B + \psi^A V_B - V^A \left(\psi_B + V^A V_B \frac{1}{2}\psi^M \psi_M\right) \in \mathrm{SO}(d,1). \tag{4.42}$$

One can check that $v^A \to O^A{}_B(v^B + \psi^B) = v^A$, while $\eta^{AB} \to O^A{}_C O^B{}_D \eta^{CD} = \eta^{AB}$ and $V^A \to O^A{}_B V^B = V^A$. On the other hand

$$e^A{}_M \to O^A{}_B e^B{}_M = e^A{}_M + \psi^A V_M - V^A \left(\psi_M - V_M \frac{1}{2}\psi^R \psi_R\right). \tag{4.43}$$

Choosing a basis and performing the null reduction, the conditions (4.41) reduce to

$$\psi_\sim = 0, \quad v^\mu \psi_\mu = -\frac{1}{2}h^{\mu\nu}\psi_\mu\psi_\nu, \qquad n_\mu \psi^\mu = 0, \quad -\psi^\sim + b_\mu \psi^\mu = -\frac{1}{2}h_{\mu\nu}\psi^\mu\psi^\nu. \tag{4.44}$$

Note that using the lowering operator and these identities we have

$$\psi_\mu = g_{\mu N}\psi^N = -n_\mu\left(\psi^\sim - b_\nu\psi^\nu\right) + h_{\mu\nu}\psi^\nu = -n_\mu \frac{1}{2}(h_{\rho\sigma}\psi^\rho\psi^\sigma) + h_{\mu\nu}\psi^\nu, \tag{4.45}$$

therefore all the information in ψ^M can be encoded into ψ^μ, which satisfies $n_\mu \psi^\mu = 0$. We find that n_μ, $h^{\mu\nu}$, and $e_a{}^\mu$ are invariant under this transformation, while

$$v^\mu \to v^\mu + \psi^\mu, \qquad b_\mu \to b_\mu + \psi_\mu, \qquad h_{\mu\nu} \to h_{\mu\nu} - 2n_{(\mu}\psi_{\nu)}, \qquad e^a{}_\mu \to e^a{}_\mu - \psi^a n_\mu. \tag{4.46a}$$

Comparing these to eq. (4.26), we can convince ourselves that we are merely performing a finite Galilean (Milne) boost parametrised by $\Lambda_\chi^{\tau\,a} = \psi^a$. Among the connections, A_μ, $\Gamma^\lambda{}_{\mu\nu}$, and $C^a{}_{\mu b}$ are invariant, while $c_{\mu a} \to c_{\mu a} + \tilde{\mathrm{D}}_\mu \psi_a$. The field strength $F_{\mu\nu}$, temporal torsion $H_{\mu\nu}$, and spatial curvature $R_{\mu\nu}{}^a{}_b$ are again invariant, while the remaining components transform as

$$\begin{gathered}\mathrm{T}^a{}_{\mu\nu} \to \mathrm{T}^a{}_{\mu\nu} + \psi^a H_{\mu\nu}, \qquad B_{\mu\nu} \to B_{\mu\nu} + \psi_a \mathrm{T}^a{}_{\mu\nu} + H_{\mu\nu}\frac{1}{2}\psi^a\psi_a, \\ R_{\mu\nu}{}^a \to R_{\mu\nu}{}^a - C^a{}_{\mu b}\psi^b.\end{gathered} \tag{4.46b}$$

We can repeat the same procedure for Noether currents as well. We get that the mass current ρ^μ, charge current j^μ, and temporal spin current $\tau^{\mu a}$ are invariant, while the other currents mix up according to

$$\epsilon^\mu \to \epsilon^\mu + \frac{1}{2}\rho^\mu \psi^a \psi_a - p^\mu{}_a \psi^a, \qquad p^\mu{}_a \to p^\mu{}_a - \rho^\mu \psi_a,$$
$$\sigma^{\mu a}{}_b \to \sigma^{\mu a}{}_b + \tau^{\mu a} \psi_b - \psi^a \tau^\mu{}_b. \tag{4.46c}$$

One can check that with these transformations, the Lorentz force f_μ defined in eq. (4.38) is rendered frame invariant.

Due to the non-trivial nature of these transformation rules, they are quite hard to implement in a purely d-dimensional description and have led much of the recent developments in the field of Newton-Cartan geometries. When working in the null background description however, this invariance is automatic because the field v^M was only introduced at the end for the purposes of reduction and is not present in the theory by itself.

4.2.4 The non-relativistic limit

Over these last two subsections, we have taken an axiomatic approach to set up Galilean invariant theories, their symmetries, background fields, and conservation equations. It is perhaps instructive to reflect back and consider how this structure could arise in a $c \to \infty$ limit of a relativistic theory. The discussion here is motivated from the work of [121]. Let us start with the d-dimensional Poincaré algebra (2.1). The first thing we should note is that we have one less generator than the Galilean algebra (4.1) that we are hoping to achieve via a non-relativistic limit. There is no analogue of the Galilean mass generator M on the relativistic side. It is quite natural because the relativistic theories do not distinguish between mass and energy; in a relativistic framework mass and energy are not independently conserved, but only their combination. To define a non-relativistic limit therefore, we must be provided with an emergent U(1) symmetry in the relativistic theory when probed at low energies. Typically, this would arise due to the emergence of certain particle number currents at low energies, which are conserved in the absence of enough energy to combine or split particles. Let us denote the generator of this additional U(1) symmetry by M, and the associated conserved current by R^μ. This symmetry does not talk to the Poincaré or G sectors of the theory, therefore M commutes with all the other generators.

To take a non-relativistic limit, we need to introduce a c-scaling for various relativistic quantities and then take $c \to \infty$. Let us start by picking a basis $(x^A) = (c\,t, x^a)$ and choose a decomposition for various Poincaré generators as

$$P_\alpha = \begin{pmatrix} -c\,M - \frac{1}{c}H + \ldots \\ P_a + \ldots \end{pmatrix}, \qquad J^\alpha{}_\beta = \begin{pmatrix} 0 & -c\,B_b + \ldots \\ -c\,B^a + \ldots & M^a{}_b + \ldots \end{pmatrix}, \tag{4.47}$$

where ellipses denote the subleading terms in c. The Poincaré commutation relations in eq. (2.1), when truncated to the highest order in c, exactly reproduce the Galilean commutators in eq. (4.1). This suggests that the Noether currents undergo a corresponding decomposition

$$T^\mu{}_\alpha = \begin{pmatrix} -c\,\rho^\mu - \frac{1}{c}\epsilon^\mu + \ldots \\ p^\mu{}_a + \ldots \end{pmatrix} \qquad \Sigma^{\mu\alpha}{}_\beta = \begin{pmatrix} 0 & -c\,\tau^\mu{}_b + \ldots \\ -c\,\tau^{\mu\,a} + \ldots & \sigma^{\mu a}{}_b + \ldots \end{pmatrix},$$
$$R^\mu = \rho^\mu + \ldots, \qquad J^\mu = j^\mu + \ldots. \tag{4.48}$$

Now let us move on to the non-relativistic limit of the relativistic background fields. In addition to the fields defined in section 2.1, we now also have a gauge field B_μ associated with the generator M. We choose the following c-scaling for these fields

$$e^\alpha{}_\mu = \begin{pmatrix} c\,n_\mu \\ e^a{}_\mu \end{pmatrix}, \quad e_\alpha{}^\mu = \begin{pmatrix} \frac{1}{c}v^\mu \\ e_a{}^\mu \end{pmatrix}, \quad B_\mu = c^2 n_\mu + b_\mu, \quad \mathrm{T}^\alpha{}_{\mu\nu} = \begin{pmatrix} -c\,H_{\mu\nu} + \frac{1}{c}\Omega_{\mu\nu} \\ \mathrm{T}^a{}_{\mu\nu} \end{pmatrix}, \tag{4.49}$$

while the gauge field A_μ stays as such. The limits of affine and spin connections can be worked out from here, which gives us exactly their Newton-Cartan expressions

$$\Gamma^\lambda{}_{\mu\nu} \to \left(\Gamma^\lambda{}_{\mu\nu}\right)_{\mathrm{NC}} + \ldots, \quad C^a{}_{\mu b} \to \left(C^a{}_{\mu b}\right)_{\mathrm{NC}} + \ldots, \quad C^a{}_{\mu t} = c_\mu{}^a + \ldots. \tag{4.50}$$

These scalings are defined based on the requirement that

$$\begin{aligned}\langle T^\mu{}_\alpha\rangle \delta e^\alpha{}_\mu + \langle \Sigma^{\mu\alpha}{}_\beta\rangle \delta C^\beta{}_{\mu\alpha} + \langle J^\mu\rangle \cdot \delta A_\mu \\ = \rho^\mu \delta b_\mu - \epsilon^\mu \delta n_\mu + p^\mu{}_a \delta e^a{}_\mu - 2\tau^\mu{}_a \delta c_\mu{}^a + \sigma^{\mu a}{}_b \delta C^b{}_{\mu a} + j^\mu \cdot \delta A_\mu + \ldots.\end{aligned} \tag{4.51}$$

This ensures that in $c \to \infty$ limit, the Galilean sources couple correctly to their respective Galilean currents. As a non-trivial check of these scaling rules, we can take a $c \to \infty$ limit of the δ_χ variations defined in eq. (2.16) and find that they indeed reduce to the Galilean ones defined in eq. (4.26), provided that we identify the spin parameters as

$$\Lambda_\chi^{\Sigma\,a}{}_t \to \Lambda_\chi^{\tau\,a}, \qquad \Lambda_\chi^{\Sigma\,a}{}_b \to \Lambda_\chi^{\sigma\,a}{}_b. \tag{4.52}$$

Finally, at the highest order in c, the relativistic conservation laws (2.20) also exactly map to the Galilean ones given in eq. (4.37).

With all this machinery in place, we are finally ready to discuss Galilean hydrodynamics. In the next section, we introduce hydrodynamics on null backgrounds and use it work out the principles of Galilean hydrodynamics coupled to Newton-Cartan backgrounds.

4.3 Null fluids

4.3.1 Hydrodynamics on null backgrounds

Following our relativistic discussion in section 2.2, we now formulate a theory of hydrodynamics on null backgrounds. We start with the assumption that the fundamental theory we are seeking to describe respects the spacetime Poincaré transformations and some internal global G-transformations, and is left invariant by the action of a null isometry $\mathcal{V}$. This postulates the existence of the associated $\mathcal{V}$ respecting conserved currents $(T^M{}_A, \Sigma^{MA}{}_B, J^M)$. In the low-energy regime, the theory would be described by some effective massless degrees of freedom φ^I. Therefore the conserved currents can generically be expressed in terms of φ^I, the background fields $(e^A{}_M, C^A{}_{MB}, A_M, \mathcal{V})$, and their derivatives. Due to Noether's theorem, these currents satisfy the identities (4.11).

Thanks to these identities, the conservation laws (4.12) can serve as a placeholder for a vector, a $\mathfrak{so}(d,1)$-valued scalar, and a $\mathfrak{g}$-valued scalar linear combinations of the equations of motion. That is to say that out of the dynamic fields φ^I, eq. (4.12) can serve as equations of motion for some degrees of freedom packaged into a set of symmetry parameters

$$\mathcal{B} = \left(\ \beta^M\ ,\ \ \Lambda^\Sigma_\beta\ ,\ \ \Lambda_\beta\ \right). \tag{4.53}$$

The remaining dynamical fields and their equations of motion are represented by φ^i and $\mathcal{E}_i \approx 0$ respectively. Recall that the spin conservation equation in eq. (4.12) is only relevant up to an arbitrary vector redefinition (4.14). Therefore we should not treat all the components in Λ^Σ_β to be independent. We can fix this by choosing $\delta_\mathcal{B} V^A = \beta^M \partial_M V^A - \Lambda^{\Sigma A}_\beta{}_B V^B = 0$, which can be checked to be a covariant condition. The fields $\mathcal{B}$ can also be represented in terms of a set of hydrodynamic fields

$$\begin{aligned}
\text{Null velocity:}\quad u^M &\quad \text{with} \quad u^M u_M = 0, \quad u^M V_M = -1, \\
\text{Temperature:}\quad T, &\qquad \text{Mass chemical potential:}\quad \mu_{\text{m}}, \\
\text{Spin chemical potential:}\quad \mu^\Sigma \in \mathfrak{so}(d,1) &\quad \text{with} \quad \mu^{\Sigma A}{}_B V^B = 0, \\
G \text{ chemical potential:}\quad \mu \in \mathfrak{g}&,
\end{aligned} \tag{4.54}$$

defined via

$$\beta^M = \frac{1}{T}\left(u^M - \mu_{\text{m}} V^M\right), \qquad \Lambda^{\Sigma A}_\beta{}_B + \beta^M C^A{}_{MB} = \frac{1}{T}\mu^{\Sigma A}{}_B, \qquad \Lambda_\beta + \beta^M A_M = \frac{1}{T}\mu. \tag{4.55}$$

We can now go ahead and write down the most generic null fluid constitutive relations: expressions for $(T^M{}_A, \Sigma^{MA}{}_B, J^M, \mathcal{E}_i)$ in terms of the dynamical fields φ^i and $\mathcal{B}$, the background fields $(e^A{}_M, C^A{}_{MB}, A_M, \mathcal{V})$, and their derivatives, arranged in a derivative expansion.

The remainder of the story exactly parallels that for relativistic fluids given in sections 2.2 and 2.3. In the following we recapitulate the key points of the discussion. Upon taking the hydrodynamic fields $\mathcal{B}$ on-shell, the partially on-shell conservation laws are given by

$$\begin{aligned}
\underline{\mathrm{D}}_M T^M{}_A &\simeq e_A{}^N \Big(\mathrm{T}^B{}_{NM} T^M{}_B + R_{NM}{}^C{}_B \Sigma^{MB}{}_C + F_{NM} \cdot J^M \Big) + \mathcal{O}^i_A \mathcal{E}_i, \\
\underline{\mathrm{D}}_M \Sigma^{MAB} &\simeq T^{[BA]} + \Sigma_{\mathrm{H}}^{\perp AB} + \mathcal{O}^{iAB} \mathcal{E}_i, \\
\underline{\mathrm{D}}_M J^M &\simeq \mathrm{J}^\perp_{\mathrm{H}} + \mathcal{O}^i \mathcal{E}_i.
\end{aligned} \tag{4.56}$$

The second law of thermodynamics requires that the null fluid constitutive relations should be accompanied by an entropy current J^M_{S} whose divergence is positive semi-definite, i.e. $\underline{\mathrm{D}}_M J^M_{\mathrm{S}} \geq 0$, for all partially on-shell (thermodynamically isolated) fluid configurations. This statement can be re-expressed in a more useful off-shell language by defining a free energy current and a free energy Hall current

$$\begin{aligned} N^M &= J^M_{\rm S} + \beta^A T^M{}_A + \left(\Lambda^{\Sigma B}_\beta{}_A + \beta^M C^B{}_{MA}\right)\Sigma^{MA}{}_B + \left(\Lambda_\beta + \beta^M A_M\right)\cdot J^M + \mathcal{N}^{iM}_\beta \mathcal{E}_i, \\ \mathrm{N}^\perp_{\rm H} &= \left(\Lambda^{\Sigma B}_\beta{}_A + \beta^M C^B{}_{MA}\right)\Sigma^{\perp a}_{\rm H}{}_b + \left(\Lambda_\beta + \beta^M A_M\right)\cdot \mathrm{J}^\perp_{\rm H}, \end{aligned} \tag{4.57}$$

where $\mathcal{N}^{iM}_\beta$ is defined in eq. (2.18). Using these definitions and the partially on-shell conservation laws (4.56), we can convert the second law statement into an adiabaticity equation

$$\underline{\mathrm{D}}_M N^M - \mathrm{N}^\perp_{\rm H} = T^M{}_A\,\delta_{\mathscr{B}} e^A{}_M + \Sigma^{MA}{}_B \delta_{\mathscr{B}} C^B{}_{MA} + J^M\cdot\delta_{\mathscr{B}} A_M + \mathcal{E}_i\delta_{\mathscr{B}}\varphi^i + \Delta, \quad \Delta\geq 0, \tag{4.58}$$

where Δ is a positive semi-definite quadratic form. The $\delta_{\mathscr{B}}$ variation of various fields is given in eq. (2.16). In writing this equation, we have already fixed a large amount of redefinition freedom in the arbitrarily defined hydrodynamic fields. The remaining freedom can be fixed by rendering our constitutive relations to be independent of $u^M\delta_{\mathscr{B}} g_{MN}$, $u^M\delta_{\mathscr{B}} C^A{}_{MB}$, and $u^M\delta_{\mathscr{B}} A_M$. In this form, the second law requires that for a set of constitutive relations $(T^M{}_A, \Sigma^{MA}{}_B, J^M, \mathcal{E}_i)$ to be physically allowed, they must admit a free energy current N^M, which satisfies the adiabaticity equation for some positive semi-definite quadratic form Δ. We classify the most generic constitutive relations allowed by this restriction in sections 4.3.3 to 4.3.5. First, let us see how the null fluid constitutive relations can be translated to the d-dimensional Galilean fluid language.

4.3.2 Null reduction to Galilean hydrodynamics

Once we are given a set of null fluid constitutive relations $(T^M{}_A, \Sigma^{MA}{}_B, J^M, \mathcal{E}_i)$, we can use the null reduction prescription outlined in section 4.2 to obtain the respective constitutive relations for a Galilean fluid. To perform the explicit reduction, however, we need a map between the dynamical fields of a null fluid and those of a Galilean fluid. Upon introduction of a frame velocity v^M, the hydrodynamic fields in eq. (4.54) can be decomposed into

$$u^\mu \quad \text{with} \quad u^\mu n_\mu = 1, \qquad T, \qquad \mu_{\rm m}, \qquad \mu^{\sigma\,a}{}_b, \qquad \mu^{\tau\,a}, \qquad \mu, \tag{4.59}$$

where

$$u^M = \begin{pmatrix} b_\mu u^\mu + \frac{1}{2}u^a u_a \\ u^\mu \end{pmatrix}, \qquad \mu^{\Sigma A}{}_B = \begin{pmatrix} 0 & 0 & \mu^\tau{}_b \\ 0 & 0 & 0 \\ 0 & \mu^{\tau\,a} & \mu^{\sigma\,a}{}_b \end{pmatrix}. \tag{4.60}$$

In terms of these, we can also find out

$$u_M = \begin{pmatrix} -1 \\ u_\mu = \bar u_\mu + b_\mu - n_\mu \frac{1}{2} u^a u_a \end{pmatrix}. \tag{4.61}$$

We have defined the "spatial" components of the Galilean fluid velocity as $\bar u^\mu = u^\mu - v^\mu$ which satisfy $\bar u^\mu n_\mu = 0$, or equivalently on the frame bundle $u^a = e^a{}_\mu u^\mu = e^a{}_\mu \bar u^\mu$. We can also define the projection operators against the fluid velocity

$$p^{\mu\nu} = h^{\mu\nu}, \qquad p_{\mu\nu} = h_{\mu\nu} - 2n_{(\mu}\bar{u}_{\nu)} + n_\mu n_\nu \bar{u}^\rho \bar{u}_\rho, \qquad p^\mu{}_\nu = p^{\mu\rho} p_{\rho\nu}. \tag{4.62}$$

The other dynamical fields φ^i in the fluid description might also admit such a decomposition, but in the absence of their explicit transformation properties, we leave them abstract in this discussion. We later look at some explicit examples in chapter 5.

Under a Galilean frame transformation $v^M \to v^M + \psi^M$ defined in section 4.2.3, most of these dynamical fields remain unchanged, except the spatial fluid velocity and temporal spin chemical potential, which transform according to

$$u^a \to u^a - \psi^a, \qquad \mu^{\tau\, a} \to \mu^{\tau\, a} + \mu^{\sigma\, a}{}_b \psi^b. \tag{4.63}$$

If instead of working in an arbitrary Galilean frame, we decided to work in the naturally defined fluid frame of reference given by $v^\mu = u^\mu$, we can choose $\psi^\mu = \bar{u}^\mu$ setting $\bar{u}^\mu \to 0$. At a calculational level therefore, it is always best to work in the fluid frame of reference. If required, we can later perform a frame transformation outlined in eq. (4.46) with $\psi^\mu = -\bar{u}^\mu$ and get back the generic frame results.

Due to the second law of thermodynamics, the null fluid constitutive relations are accompanied by an entropy current J^M_{S} and a quadratic form Δ, such that for thermodynamically isolated fluid configurations, we have $\underline{\mathrm{D}}_M J^M_{\mathrm{S}} \simeq \Delta \geq 0$. Upon null reduction, this statement transforms into a local second law for Galilean fluids,

$$\tilde{\underline{\mathrm{D}}}_\mu s^\mu \simeq \Delta \geq 0, \qquad \text{where} \quad s^\mu = J^\mu_{\mathrm{S}}. \tag{4.64}$$

Following our null fluid discussion, we know that to satisfy the second law requirement, the Galilean fluid must admit a free energy current $n^\mu = N^\mu$ such that the following adiabaticity equation is satisfied for any (on-shell or off-shell) fluid configuration

$$\begin{aligned}\tilde{\underline{\mathrm{D}}}_\mu n^\mu - \mathrm{n}^\perp_{\mathrm{H}} = \rho^\mu \delta_{\mathscr{B}} b_\mu - \epsilon^\mu \delta_{\mathscr{B}} n_\mu + p^\mu{}_a \delta_{\mathscr{B}} e^a{}_\mu \\ - 2\tau^{\mu a} \delta_{\mathscr{B}} c_{\mu a} + \sigma^{\mu a}{}_b \delta_{\mathscr{B}} C^b{}_{\mu a} + j^\mu \cdot \delta_{\mathscr{B}} A_\mu + \mathcal{E}_i \delta_{\mathscr{B}} \varphi^i + \Delta.\end{aligned} \tag{4.65}$$

Here $\mathrm{n}^\perp_{\mathrm{H}} = \mathrm{N}^\perp_{\mathrm{H}}$ is the anomalous free energy Hall current. The $\delta_{\mathscr{B}}$ variation of various fields used here can be obtained by a substituting $\mathscr{X}$ with $\mathscr{B}$ in eq. (4.26).

To summarise, a Galilean fluid is characterised by its conserved currents, which in the minimal setting are the mass current ρ^μ, energy current ϵ^μ, and momentum current $p^\mu{}_a$. Depending on the physical application we have in mind, we can also throw in some possibly non-Abelian charge currents j^μ, spin currents $\tau^{\mu a}$ and $\sigma^{\mu ab}$, and equations of motion $\mathcal{E}_i$ corresponding to additional gapless modes in the fluid description (like Goldstone modes of broken symmetries). We start with the most generic form of these currents allowed by the Galilean symmetries, called the constitutive relations, in terms of some dynamical fields: fluid velocity u^μ, temperature T, chemical potentials μ_{m}, $\mu^{\tau\, a}$, $\mu^{\sigma\, ab}$, and μ, and additional gapless fields φ^i, arranged in a derivative expansion. Dynamical equations for these fields are given by the conservation equations (4.37). For convenience, we can also introduce some background fields: frame fields $e^a{}_\mu$, $e_a{}^\mu$, and n_μ, connections $\Gamma^\lambda{}_{\mu\nu}$, $C^a{}_{\mu b}$, A_μ, and b_μ, and a Galilean frame velocity v^μ. To be physically relevant, the fluid constitutive relations are required to satisfy the second law of thermodynamics, which essentially requires them to be accompanied by an entropy current s^μ whose divergence is locally positive semi-definite. This can equivalently be posed as the requirement that there must exist a free energy current n^μ so that the adiabaticity equation (4.65) is satisfied. To work out these constitutive relations in a manner that manifestly preserves the Galilean invariance, it is convenient to work in a one-higher dimensional null fluid picture, where the symmetry structure is exactly that of a relativistic fluid. The Galilean adiabaticity equation also happens to be equivalent to the adiabaticity equation in the null fluid formalism, which can be used to derive the constraints on the Galilean fluid constitutive relations in a more convenient language.

4.3.3 Anomaly induced transport

Having derived the Galilean version of the adiabaticity equation, we would now like to inspect and classify all of its solutions. Since the anomaly inflow mechanism on null backgrounds, as discussed in section 4.1.4, is subtly different from its relativistic counterpart, in this subsection we first focus on a particular anomaly induced solution to eq. (4.58). Later in section 4.3.4, we also discuss its extension to include transcendental anomalies.

Following [5], we note that the shadow connections for null fluids should be defined with respect to the null isometry V^M, as opposed to the fluid velocity which was used for relativistic fluids in eq. (2.58), i.e.

$$\hat{\boldsymbol{A}} = \boldsymbol{A} + \mu\,\boldsymbol{V}, \qquad \hat{\boldsymbol{C}}^A{}_B = \boldsymbol{C}^A{}_B + \mu^{\Sigma\,A}{}_B\,\boldsymbol{V}, \tag{4.66}$$

where $\boldsymbol{V} = V_M \mathrm{d}x^M$. They satisfy

$$\Lambda_\beta + \beta^M \hat{A}_M = \Lambda^{\Sigma\,A}_\beta{}_B + \beta^M \hat{C}^A{}_{MB} = 0, \qquad \Lambda_V + V^M \hat{A}_M = \Lambda^{\Sigma\,A}_V{}_B + V^M \hat{C}^A{}_{MB} = 0. \tag{4.67}$$

We can use these to work out the respective shadow field strength and curvature, followed by the shadow anomaly polynomial and Hall currents. In particular, the shadow free energy Hall current is given by

$$\star_{(d+2)}\hat{\mathbf{N}}_\mathrm{H} = \frac{1}{T}\star_{(d+2)}\Big(\mu\cdot\hat{\mathbf{J}}_\mathrm{H} + \mu^\Sigma\cdot\hat{\boldsymbol{\Sigma}}_\mathrm{H}\Big). \tag{4.68}$$

Let us further define

$$\begin{gathered}\star\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{P}}} = \frac{\boldsymbol{V}}{\mathrm{d}\boldsymbol{V}}\wedge\star_{(d+2)}\Big(\boldsymbol{\Sigma}_\mathrm{H} - \hat{\boldsymbol{\Sigma}}_\mathrm{H}\Big), \qquad \star\boldsymbol{J}_{\boldsymbol{\mathcal{P}}} = \frac{\boldsymbol{V}}{\mathrm{d}\boldsymbol{V}}\wedge\star_{(d+2)}\Big(\mathbf{J}_\mathrm{H} - \hat{\mathbf{J}}_\mathrm{H}\Big), \\ \star\boldsymbol{q}_{\boldsymbol{\mathcal{P}}} = -\frac{\boldsymbol{V}}{\mathrm{d}\boldsymbol{V}\wedge\mathrm{d}\boldsymbol{V}}\wedge\Big(\boldsymbol{\mathcal{P}} - \hat{\boldsymbol{\mathcal{P}}} + T\,\mathrm{d}\boldsymbol{V}\wedge\star_{(d+2)}\hat{\mathbf{N}}_\mathrm{H}\Big).\end{gathered} \tag{4.69}$$

In terms of these, we can read out the anomaly induced constitutive relations, also known as the Class A transport, as simply

$$(T^M{}_A)_\mathrm{A} = q^M_{\boldsymbol{\mathcal{P}}} V_A, \qquad (\Sigma^{MA}{}_B)_\mathrm{A} = \Sigma_{\boldsymbol{\mathcal{P}}}{}^A{}_B, \qquad (J^M)_\mathrm{A} = J^M_{\boldsymbol{\mathcal{P}}}. \tag{4.70a}$$

along with $(\mathcal{E}_i)_{\mathrm{A}} = 0$. They satisfy the null fluid adiabaticity equation (4.58) with the free energy current

$$(N^M)_{\mathrm{A}} = \frac{1}{T}\left(-q^M_{\boldsymbol{\mathcal{P}}} + \mu^{\Sigma\,A}{}_B\,\Sigma^{MB}_{\boldsymbol{\mathcal{P}}}{}_A + \mu\cdot J^M_{\boldsymbol{\mathcal{P}}}\right). \tag{4.70b}$$

Like in the relativistic case, we note that there is no Class A contribution to the entropy current J^M_{S}. Upon performing the null reduction, we can see that, in fact, only the energy current ϵ^μ, spatial spin current $\sigma^{\mu ab}$, charge current j^μ, and free energy current n^μ get an anomaly-induced contribution. All the other currents stay untouched by anomalies.

That the solutions (4.70) satisfy the Galilean adiabaticity equation (4.58), can either be established via an explicit computation, or more easily via a bulk effective action

$$S_{\mathrm{A}} = \int_{\mathcal{B}} \frac{\boldsymbol{V}}{\mathrm{d}\boldsymbol{V}} \wedge \left(\boldsymbol{\mathcal{P}} - \hat{\boldsymbol{\mathcal{P}}}\right), \tag{4.71}$$

which generates the anomaly induced constitutive relations at the boundary. The adiabaticity equation follows from here by invoking the invariance of this action under an infinitesimal symmetry variation along $\mathcal{B}$.

4.3.4 Transcendental anomalies

We can extend the Class A constitutive relations (4.70) to include slightly more generic solutions of the adiabaticity equation (4.58). To this end, like we did in section 2.3.3, we extend our theory by introducing an auxiliary $\mathrm{U}(1)_{\mathsf{T}}$ global symmetry, with background gauge field A^{T}_M and chemical potential $\mu^{\mathsf{T}} = T(\Lambda^{\mathsf{T}}_\beta + \beta^M A^{\mathsf{T}}_M) = T$. The associated shadow gauge field is given by $\hat{\boldsymbol{A}}^{\mathsf{T}} = \boldsymbol{A}^{\mathsf{T}} + T\boldsymbol{V}$. Having done that, we can use the respective field strength F^{T}_{MN} to write down new terms in the anomaly polynomial. Unlike the relativistic case, however, we can further involve terms in the anomaly polynomial made out of the Chern classes of $\boldsymbol{\Omega} = \mathrm{d}\boldsymbol{u}$. The shadow field corresponding to $\boldsymbol{u}$ is given by $\hat{\boldsymbol{u}} = \boldsymbol{u} + \mu_{\mathrm{m}}\boldsymbol{V}$, with the property that $\beta^M \hat{u}_M = 0$. There is no analogue of this in relativistic hydrodynamics. Finally, the full thermal anomaly polynomial for this enlarged theory is given as

$$\boldsymbol{\mathcal{P}}_{\mathsf{T}} = \boldsymbol{\mathcal{P}} + \boldsymbol{\mathcal{P}}_{\mathrm{H_V}}, \qquad \boldsymbol{\mathcal{P}}_{\mathrm{H_V}} = \boldsymbol{u} \wedge \sum_{j\geq 1} (\boldsymbol{F}^{\mathsf{T}})^{\wedge j} \wedge \boldsymbol{\mathcal{P}}_{\mathrm{H_V},j}, \tag{4.72}$$

where $\boldsymbol{\mathcal{P}}$ is the original anomaly polynomial made out of $\boldsymbol{F}$ and $\boldsymbol{R}$, while $\boldsymbol{\mathcal{P}}_{\mathrm{H_V},j}$ are $(d+2-2j)$-rank anomaly polynomials which can in addition involve $\boldsymbol{\Omega}$. Note that since we now have a preferred vector field u^M, we have used it to define the anomaly polynomials instead of an arbitrary vector field v^M used in section 4.1.4.

Now, we can use the mechanism outlined in the previous subsection to generate a particular solution for the adiabaticity equation of the enlarged theory

$$
\begin{aligned}
&\underline{\mathrm{D}}_M N^M - \mathrm{N}^\perp_{\mathrm{H}} - \mathrm{J}^\perp_{\mathsf{T}\mathrm{H}} \\
&\quad = T^M{}_A\, \delta_{\mathscr{B}} e^A{}_M + \Sigma^{MA}{}_B \delta_{\mathscr{B}} C^B{}_{MA} + J^M \cdot \delta_{\mathscr{B}} A_M + J^M_{\mathsf{T}} \delta_{\mathscr{B}} A^{\mathsf{T}}_M + J^M_\Omega \delta_{\mathscr{B}} u_M + \mathcal{E}_i \delta_{\mathscr{B}} \varphi^i \\
&\quad = \Big(T^M{}_A + J^N_\Omega P_{NA} u^M + J^M_\Omega u_A \Big) \delta_{\mathscr{B}} e^A{}_M \\
&\qquad + \Sigma^{MA}{}_B \delta_{\mathscr{B}} C^B{}_{MA} + J^M \cdot \delta_{\mathscr{B}} A_M + J^M_{\mathsf{T}} \delta_{\mathscr{B}} A^{\mathsf{T}}_M + \mathcal{E}_i \delta_{\mathscr{B}} \varphi^i .
\end{aligned}
\tag{4.73}
$$

In the second step, we have expanded $\delta_{\mathscr{B}} u_M = (P_{MA} u^N + \delta^N{}_M u_A)\, \delta_{\mathscr{B}} e^A{}_N$. Various Hall currents are now being defined with respect to the thermal anomaly polynomial $\boldsymbol{\mathcal{P}}_{\mathsf{T}}$, in particular

$$
\begin{aligned}
\star_{(d+2)} \mathbf{J}_{\mathsf{T}\mathrm{H}} &= \frac{\partial \boldsymbol{\mathcal{P}}_{\mathsf{T}}}{\partial \boldsymbol{F}^{\mathsf{T}}} = \boldsymbol{u} \wedge \sum_{j\geq 1} j (\boldsymbol{F}^{\mathsf{T}})^{\wedge (j-1)} \wedge \boldsymbol{\mathcal{P}}_{\mathrm{H_V},j}, \\
\star_{(d+2)} \mathbf{J}_{\Omega\mathrm{H}} &= \frac{\partial \boldsymbol{\mathcal{P}}_{\mathsf{T}}}{\partial \boldsymbol{\Omega}} = \boldsymbol{u} \wedge \sum_{j\geq 1} (\boldsymbol{F}^{\mathsf{T}})^{\wedge j} \wedge \frac{\partial \boldsymbol{\mathcal{P}}_{\mathrm{H_V},j}}{\partial \boldsymbol{\Omega}} .
\end{aligned}
\tag{4.74}
$$

On the other hand, we have defined the free energy Hall current to be

$$
\star_{(d+2)} \mathbf{N}_{\mathrm{H}} = \frac{1}{T} \star_{(d+2)} \left(\mu^{\Sigma\, A}{}_B \boldsymbol{\Sigma}_{\mathrm{H}}{}^B{}_A + \mu \cdot \mathbf{J}_{\mathrm{H}} + \mu_{\mathrm{m}} \mathbf{J}_{\Omega\mathrm{H}} \right). \tag{4.75}
$$

In the limit that we take $F^{\mathsf{T}}_{MN} \to 0$, $\mathbf{J}_{\Omega\mathrm{H}}$ vanishes and $\mathbf{N}_{\mathrm{H}}$ takes its original unextended form. The quantity $\delta_{\mathscr{B}} A^{\mathsf{T}}_M = \beta^N F^{\mathsf{T}}_{NM}$ also vanishes. With these simplifications, we can show that the extended adiabaticity equation reduces to

$$
\begin{aligned}
&\underline{\mathrm{D}}_M N^M - \mathrm{J}^\perp_{\mathsf{T}\mathrm{H}} - \mathrm{N}^\perp_{\mathrm{H}} \\
&\quad = \Big(T^M{}_A + J^N_\Omega P_{NA} u^M + J^M_\Omega u_A \Big) \delta_{\mathscr{B}} e^A{}_M + \Sigma^{MA}{}_B \delta_{\mathscr{B}} C^B{}_{MA} + J^M \cdot \delta_{\mathscr{B}} A_M + \mathcal{E}_i \delta_{\mathscr{B}} \varphi^i ,
\end{aligned}
\tag{4.76}
$$

This is almost the form of the adiabaticity equation that we require for our original theory, except for the residual $\star_{(d+2)} \mathbf{J}_{\mathsf{T}\mathrm{H}} = \boldsymbol{u} \wedge \boldsymbol{\mathcal{P}}_{\mathrm{H_V},1}$ piece. Note that $\star_{(d+2)} \mathbf{J}_{\mathsf{T}\mathrm{H}}$ is closed, as its derivative does not have any component along V^M. If it also happens to be exact, we can absorb it into the free energy current. To this end, we can use the fact that there exists a gauge-non-invariant $\boldsymbol{I}_{\mathrm{H_V},1}$ such that $\mathrm{d}\boldsymbol{I}_{\mathrm{H_V},1} = \boldsymbol{\mathcal{P}}_{\mathrm{H_V},1}$, and therefore

$$
\star_{(d+2)} \mathbf{J}_{\mathsf{T}\mathrm{H}} = \boldsymbol{u} \wedge \mathrm{d}\boldsymbol{I}_{\mathrm{H_V},1} = -\mathrm{d}\left(\boldsymbol{u} \wedge \boldsymbol{I}_{\mathrm{H_V},1} \right) - \boldsymbol{u} \wedge \boldsymbol{\Omega} \wedge \iota_V \boldsymbol{I}_{\mathrm{H_V},1}. \tag{4.77}
$$

Here ι_V denotes the interior product along V^M. If we work in the transverse gauge for the null isometry, i.e. $\Lambda_V = \Lambda^\Sigma_V = 0$, where $\iota_V \boldsymbol{A} = \iota_V \boldsymbol{C} = 0$, we can always ensure the last piece to vanish as long as $\boldsymbol{\mathcal{P}}_{\mathrm{H_V},1}$ has at least one instance of $\boldsymbol{F}$ or $\boldsymbol{R}$. In that case, $\star_{(d+2)} \mathbf{J}_{\mathsf{T}\mathrm{H}}$ is, in fact, exact. However, if it is purely made out of $\boldsymbol{\Omega}$, i.e. $\boldsymbol{\mathcal{P}}_{\mathrm{H_V},1} = \boldsymbol{\Omega}^{\wedge d/2}$, the respective term decisively does not vanish

$$
\boldsymbol{u} \wedge \boldsymbol{\Omega} \wedge \iota_V \boldsymbol{I}_{\mathrm{H_V},1} = -\boldsymbol{u} \wedge \boldsymbol{\Omega}^{\wedge d/2} \neq 0. \tag{4.78}
$$

Consequently, to generate constitutive relations for our Galilean system of interest, we are not allowed to have the term $\boldsymbol{u} \wedge \boldsymbol{F}_{\mathsf{T}} \wedge \boldsymbol{\Omega}^{\wedge d/2}$ in the thermal anomaly polynomial $\boldsymbol{\mathcal{P}}_{\mathsf{T}}$. With this taken care of, we can rewrite eq. (4.76) into

$$\begin{aligned}\underline{\mathrm{D}}_M \left(N^M - \star_{(d+1)} \left(\boldsymbol{u} \wedge \boldsymbol{I}_{\mathrm{H_V},1}\right)^M\right) - \mathrm{N}^\perp_{\mathrm{H}} \\ = \left(T^M{}_A + J^N_\Omega P_{NA} u^M + J^M_\Omega u_A\right) \delta_{\mathscr{B}} e^A{}_M + \Sigma^{MA}{}_B \delta_{\mathscr{B}} C^B{}_{MA} + J^M \cdot \delta_{\mathscr{B}} A_M + \mathcal{E}_i \delta_{\mathscr{B}} \varphi^i, \quad (4.79)\end{aligned}$$

which is precisely the form that we expect for our original theory. Therefore, we can use the particular solutions of the extended adiabaticity equation to generate new solutions of our original adiabaticity equation.

Let us now return to the actual solutions. Following the procedure of the previous subsection, we note that the $\boldsymbol{\mathcal{P}}$ part of the thermal anomaly polynomial $\boldsymbol{\mathcal{P}}_{\mathsf{T}}$ simply generates the Class A constitutive relations as before. On the other hand, the $\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}$ part generates the new Class $\mathrm{H_V}$ constitutive relations. Let us denote the Hall currents defined using $\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}$ with the wrapper $(\ )_{\mathrm{H_V}}$. This allows us to define the analogues of eq. (4.69) for Class $\mathrm{H_V}$ in $F^{\mathsf{T}}_{MN} \to 0$ limit

$$\begin{gathered}\star\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} = -\frac{\boldsymbol{V}}{\mathrm{d}\boldsymbol{V}} \wedge \star_{(d+2)} (\hat{\boldsymbol{\Sigma}}_{\mathrm{H}})_{\mathrm{H_V}}\Big|_{\boldsymbol{F}^{\mathsf{T}}\to 0}, \qquad \star\boldsymbol{J}_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} = -\frac{\boldsymbol{V}}{\mathrm{d}\boldsymbol{V}} \wedge \star_{(d+2)} (\hat{\mathbf{J}}_{\mathrm{H}})_{\mathrm{H_V}}\Big|_{\boldsymbol{F}^{\mathsf{T}}\to 0}, \\ \star\boldsymbol{J}_{\mathsf{T}\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} = \frac{\boldsymbol{V}}{\mathrm{d}\boldsymbol{V}} \wedge \star_{(d+2)} \left((\mathbf{J}_{\mathsf{T}\mathrm{H}})_{\mathrm{H_V}} - (\hat{\mathbf{J}}_{\mathsf{T}\mathrm{H}})_{\mathrm{H_V}}\right)\Big|_{\boldsymbol{F}^{\mathsf{T}}\to 0}, \\ \star\boldsymbol{J}_{\Omega\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} = -\frac{\boldsymbol{V}}{\mathrm{d}\boldsymbol{V}} \wedge \star_{(d+2)} (\hat{\mathbf{J}}_{\Omega\mathrm{H}})_{\mathrm{H_V}}\Big|_{\boldsymbol{F}^{\mathsf{T}}\to 0} \\ \star\boldsymbol{q}_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} = \frac{\boldsymbol{V}}{(\mathrm{d}\boldsymbol{V})^{\wedge 2}} \wedge \left(\hat{\boldsymbol{\mathcal{P}}}_{\mathrm{H_V}} - T\mathrm{d}\boldsymbol{V} \wedge \star_{(d+2)} \left((\hat{\mathbf{N}}_{\mathrm{H}})_{\mathrm{H_V}} + (\hat{\mathbf{J}}_{\mathsf{T}\mathrm{H}})_{\mathrm{H_V}}\right)\right)\Big|_{\boldsymbol{F}^{\mathsf{T}}\to 0}. \qquad (4.80)\end{gathered}$$

In terms of these, the Class $\mathrm{H_V}$ constitutive relations are given as

$$\begin{gathered}(T^M{}_A)_{\mathrm{H_V}} = q^M_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} V_A + J^M_{\Omega\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} u_A + J^N_{\Omega\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} P_{NA} u^M, \\ (\Sigma^{MAB})_{\mathrm{H_V}} = \Sigma^{MAB}_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}}, \qquad (J^M)_{\mathrm{H_V}} = J^M_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}}, \\ (N^M)_{\mathrm{H_V}} = J^M_{\mathsf{T}\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} + \frac{1}{T}\left(-q^M_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} + \mu^\Sigma \cdot \Sigma^M_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} + \mu \cdot J^M_{\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}} + \mu_{\mathrm{m}} J^M_{\Omega\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}}\right) - \star_{(d+1)} \left(\boldsymbol{u} \wedge \boldsymbol{I}_{\mathrm{H_V},1}\right)^M,\end{gathered} \qquad (4.81)$$

along with $(\mathcal{E}_i)_{\mathrm{H_V}} = 0$. The associated entropy current, like the relativistic case, is non-trivial in Class $\mathrm{H_V}$

$$(J^M_{\mathrm{S}})_{\mathrm{H_V}} = J^M_{\mathsf{T}\boldsymbol{\mathcal{P}}_{\mathrm{H_V}}}. \qquad (4.82)$$

In fact, in contrast to Class A, all the currents, except the temporal spin current $\tau^{\mu a}$, get a contribution form Class $\mathrm{H_V}$.

The Class $\mathrm{H_V}$ constitutive relations satisfy the non-anomalous version of the Galilean adiabaticity equation (4.58). It can be verified by noting that these constitutive relations can be generated from the boundary variation of a bulk effective action

$$S_{\mathrm{H_V}} = -\int_{\mathcal{B}} \frac{\boldsymbol{V}}{\mathrm{d}\boldsymbol{V}} \wedge \hat{\boldsymbol{\mathcal{P}}}_{\mathrm{H_V}}\Big|_{\boldsymbol{F}^{\mathsf{T}}\to 0}. \qquad (4.83)$$

The non-anomalous adiabaticity equation follows by making use of the invariance of this effective action under an infinitesimal variation along $\mathscr{B}$.

Before closing the subsection, let us make an interesting observation about the torsion-less case. Because, $2\partial_{[M}V_{N]} = \mathrm{T}^A{}_{MN}V_A$, the 2-form $\mathrm{d}\boldsymbol{V} = 0$ on torsion-less backgrounds. Also, $\boldsymbol{V} \wedge \hat{\boldsymbol{F}}^{\mathsf{T}} = T\boldsymbol{V} \wedge \mathrm{d}\boldsymbol{V} = 0$. It follows that all the terms in the extended anomaly polynomial that have more than two instances of $\boldsymbol{F}^{\mathsf{T}}$ (i.e. $j > 2$) do not contribute to the Class $\mathrm{H_V}$ constitutive relations at all. So, for torsionless null hydrodynamics, the Class $\mathrm{H_V}$ anomaly polynomial simplifies to

$$\boldsymbol{\mathcal{P}}_{\mathrm{H_V}} = \boldsymbol{u} \wedge \boldsymbol{F}^{\mathsf{T}} \wedge \boldsymbol{\mathcal{P}}_{\mathrm{H_V},1} + \boldsymbol{u} \wedge \boldsymbol{F}^{\mathsf{T}} \wedge \boldsymbol{F}^{\mathsf{T}} \wedge \boldsymbol{\mathcal{P}}^{(d-2)}_{\mathrm{H_V},2}, \tag{4.84}$$

which is characterised by far less number of constants than the full anomaly polynomial. This should be contrasted with the relativistic case, where no such simplification happens in the torsion-less limit.

4.3.5 Classification of hydrodynamic transport

Having discussed two particular classes of solutions, we are now ready to exhaust the most generic solutions of the Galilean adiabaticity equation. As the analysis is an exact analogue of our relativistic discussion in section 2.3, we will be brief. Denoting the hydrodynamic constitutive relations as $\mathcal{C}$ and the non-hydrodynamic fields as Φ given in eq. (2.54), which are seen as vectors in a hybrid vector space $\mathfrak{V}$, the Galilean adiabaticity equation (4.58) can be expressed in a compact form

$$\underline{\mathrm{D}}_M N^M = \mathrm{N}^{\perp}_{\mathrm{H}} + \mathcal{C} \cdot \delta_{\mathscr{B}}\Phi + \Delta, \qquad \Delta \geq 0. \tag{4.85}$$

As this equation has the exact same structure as its relativistic counterpart in eq. (2.55), we can follow through the same discussion for classifying all of its solutions. To summarise, the hydrodynamic constitutive relations satisfying eq. (4.85) can be classified into six distinct classes:

- **Hydrostatic sector:** In this sector, constitutive relations and free energy current are constructed out of independent tensor structures that, or any of their linear combinations, do not vanish in a hydrostatic configuration. That is to say that they cannot contain an instance of "$\delta_{\mathscr{B}}$". They are classified into three classes:

  1. **Class A (anomaly induced transport):** These constitutive relations are induced by the anomalies in our symmetries. They are completely fixed in terms of the anomaly polynomial, which in turn is characterised by a set of constant anomaly coefficients. See section 4.3.3.
  2. **Class $\mathrm{H_V}$ (hydrostatic vector transport):** These are the non-anomalous hydrostatic constitutive relations whose corresponding free-energy current flows transverse to β^M and V^M. They are also completely determined up to some arbitrary constants. See section 4.3.4.
  3. **Class $\mathrm{H_S}$ (hydrostatic scalar transport):** These are the hydrostatic constitutive relations whose free energy current has a component that flows along β^M.

They are characterised by a hydrostatic scalar density $\mathcal{N}$ via the formula given in eq. (2.74). See section 2.3.4.

- **Non-hydrostatic sector:** In this sector, constitutive relations and free energy current are made out of independent tensor structures that vanish in a hydrostatic configuration. That is to say that they involve at least one instance of "$\delta_{\mathfrak{B}}$". They are classified into two classes:

4. **Class $\overline{\mathrm{D}}$ (non-dissipative transport):** These are the non-hydrostatic constitutive relations that do not cause any production of entropy. They are completely determined by $\overline{\mathfrak{D}}_n|_{n\geq 0} \in \mathfrak{V}\times\mathfrak{V}$ with $\overline{\mathfrak{D}}_n^{\mathrm{T}} = -(-)^n\overline{\mathfrak{D}}_n$. See section 2.3.5.
5. **Class D (dissipative transport):** These are the constitutive relations which are solely responsible for the production of entropy. They are completely determined by $\mathfrak{D}_n|_{n\geq 0} \in \mathfrak{V}\times\mathfrak{V}$ with $\overline{\mathfrak{D}}_n^{\mathrm{T}} = (-)^n\overline{\mathfrak{D}}_n$. See section 2.3.5.

- Finally, we also have a set of trivial solutions to the adiabaticity equation

6. **Class S (entropy transport):** This class contains solutions of the adiabaticity equation with vanishing constitutive relations but non-trivial free energy and entropy transport. These include $N^M \sim \mathcal{N}_V V^M + \underline{\mathrm{D}}_N X^{MN} + \frac{1}{2}\mathrm{T}^M{}_{NR}X^R$ for a scalar $\mathcal{N}_V$ and an antisymmetric tensor X^{MN}, for which $\underline{\mathrm{D}}_M N^M$ is trivially zero. The remainder of the class is completely characterised by a set of matrices $\mathfrak{S}_{mn}|_{m,n\geq 1}$ with $\mathfrak{S}_{mn}^{\mathrm{T}} = \mathfrak{S}_{nm}$. See section 2.3.6.

The Class S constitutive relations are not genuine hydrodynamic transport. They merely characterise the multitude of entropy currents which satisfy the second law for the same set of constitutive relations.

This finishes our discussion of the null/Galilean fluid constitutive relations and their classification following from the second law of thermodynamics. In the next chapter, we study some specific examples of Galilean hydrodynamics to illustrate how this construction works.

5 | Applications: Galilean hydrodynamics

In chapter 4 we formulated a new framework to study Galilean hydrodynamics. In this chapter, we explore how it works in practice. We first study ordinary Galilean fluids at one-derivative order and later extend to Galilean superfluids and Galilean fluids with surfaces. For concreteness, we focus on $d = 4$, which means that our null fluids are 5-dimensional, but Galilean fluids are 4-dimensional.

5.1 | Galilean fluids

Our first example are the ordinary Galilean fluids which we derived heuristically in section 1.1.3. They do not possess any additional gapless modes, neither do that have any spin-current or torsion. The absence of background torsion requires us to set $\partial_{[M} V_{N]} = 0$ in our final results. These Galilean fluids are the most standard examples of Galilean hydrodynamics, or hydrodynamics at large. The results obtained here can, therefore, be verified with a standard text like [10].

5.1.1 Ideal null fluids

Let us start with the zero derivative order ideal null fluids. Class $\mathrm{H_S}$ is the only non-empty class at this order, which is characterised by a hydrostatic scalar density $\mathcal{N}$, given by an arbitrary scalar $P(T, \mu, \mu_{\mathrm{m}})$ as a function of the temperature T, chemical potential μ, and mass chemical potential μ_{m} of the fluid. The $\delta_{\mathscr{B}}$ variations of these ideal order scalars are given as

$$\delta_{\mathscr{B}} T = T V^{(M} u^{N)} \delta_{\mathscr{B}} g_{MN}, \quad \delta_{\mathscr{B}} \frac{\mu_{\mathrm{m}}}{T} = \frac{1}{2T} u^M u^N \delta_{\mathscr{B}} g_{MN}, \quad \delta_{\mathscr{B}} \frac{\mu}{T} = \frac{1}{T} u^M \delta_{\mathscr{B}} A_M. \tag{5.1}$$

Using these, we can compute the divergence of the ideal order free energy current to be

$$\begin{aligned} \mathrm{D}_\mu \left(P \beta^M \right) &= \frac{1}{\sqrt{-g}} \delta_{\mathscr{B}} \left(\sqrt{-g}\, P \right) = \frac{1}{2} P g^{MN} \delta_{\mathscr{B}} g_{MN} + \frac{\partial P}{\partial T} \delta_{\mathscr{B}} T + \frac{\partial P}{\partial \mu_{\mathrm{m}}} \delta_{\mathscr{B}} \mu_{\mathrm{m}} + \frac{\partial P}{\partial \mu} \cdot \delta_{\mathscr{B}} \mu, \\ &= \Big(R u^M u^N + 2 E u^{(M} V^{N)} + P P^{MN} \Big) \frac{1}{2} \delta_{\mathscr{B}} g_{MN} + Q u^M \cdot \delta_{\mathscr{B}} A_M, \end{aligned} \tag{5.2}$$

where $P^{MN} = g^{MN} + 2 u^{(M} V^{N)}$ and we have defined

$$S = \frac{\partial P}{\partial T}, \qquad R = \frac{\partial P}{\partial \mu_{\mathrm{m}}}, \qquad Q = \frac{\partial P}{\partial \mu}, \qquad E = T \frac{\partial P}{\partial T} + \mu_{\mathrm{m}} \frac{\partial P}{\partial \mu_{\mathrm{m}}} + \mu \cdot \frac{\partial P}{\partial \mu} - P. \tag{5.3}$$

Comparing eq. (5.2) with eq. (2.74), we can easily read out the ideal null fluid constitutive relations, free energy, and entropy currents

$$
\begin{aligned}
T^{MN} &= Ru^M u^N + 2Eu^{(M}V^{N)} + PP^{MN} + \mathcal{O}(\partial), \qquad J^M = Qu^M + \mathcal{O}(\partial), \\
N^M &= \frac{1}{T} P u^M + \mathcal{O}(\partial), \qquad J^M_{\mathrm{S}} = S u^M + \mathcal{O}(\partial).
\end{aligned} \tag{5.4}
$$

Using eq. (5.3), we can also show that the coefficients appearing here follow a set of thermodynamic relations

$$
\begin{aligned}
&\text{Gibbs-Duhem equation:} && \mathrm{d}P = S\,\mathrm{d}T + R\,\mathrm{d}\mu_{\mathrm{m}} + Q\cdot \mathrm{d}\mu, \\
&\text{Euler scaling relation:} && E + P = S\,T + R\,\mu_{\mathrm{m}} + Q\cdot \mu, \\
&\text{First law of thermodynamics:} && \mathrm{d}E = T\,\mathrm{d}S + \mu_{\mathrm{m}}\mathrm{d}R + \mu\cdot \mathrm{d}Q.
\end{aligned} \tag{5.5}
$$

Similar to the ideal relativistic fluids, we see that the ideal null fluids also are completely characterised by their equation of state $P = P(T, \mu, \mu_{\mathrm{m}})$. We have arrived at the null fluid constitutive relations that we derived in section 1.1.3.

Having worked out the null fluid constitutive relations, we can perform the null reduction prescribed in section 4.3.2 and read out the respective ideal Galilean fluid constitutive relations in an arbitrary frame of reference. We find

$$
\begin{aligned}
\epsilon^\mu &= \left(E + \frac{1}{2} R u^a u_a\right) u^\mu + P\bar{u}^\mu + \mathcal{O}(\partial), \qquad p^\mu{}_a = Ru^\mu u_a + P e_a{}^\mu + \mathcal{O}(\partial), \\
\rho^\mu &= Ru^\mu + \mathcal{O}(\partial), \qquad j^\mu = Qu^\mu + \mathcal{O}(\partial), \\
n^\mu &= \frac{1}{T} P u^\mu + \mathcal{O}(\partial), \qquad s^\mu = S u^\mu + \mathcal{O}(\partial).
\end{aligned} \tag{5.6}
$$

From here we can identify E as the energy density, R as the mass density, P as the isotropic pressure, Q as the charge density, and S as the entropy density of the Galilean fluid. Note that $n_\mu p^\mu{}_a = \rho^\mu e_{a\mu} = Ru_a$, therefore the boost Ward identity in eq. (4.37) is trivially satisfied with $\tau^{\mu a} = 0$. The same is true for the rotation Ward identity as well, because the stress tensor is symmetric, i.e. $p^{\mu[a} e^{b]}{}_\mu = 0$. This is characteristic of torsionless Galilean hydrodynamics.

5.1.2 One-derivative corrections

We are interested in the possible one-derivative corrections that these constitutive relations can admit. With this in mind, let us make some definitions which are useful in the following discussion

$$
\begin{aligned}
\Theta &= \mathrm{D}_M u^M, \qquad \sigma^{MN} = P^{M\langle R}P^{S\rangle N}\mathrm{D}_M u_N = P^{MR}P^{NS}\left(\mathrm{D}_{(M}u_{N)} - \frac{1}{3}P_{RS}\Theta\right), \\
\mathfrak{a}^M &= u^N \mathrm{D}_N u^M, \qquad \omega^M = \epsilon^{MNRST} V_N u_R \partial_S u_T, \\
E^M &= F^{MN} u_N, \qquad B^M = \frac{1}{2}\epsilon^{MNRST} V_N u_R F_{ST}.
\end{aligned} \tag{5.7}
$$

$\boldsymbol{\mathcal{P}}_{\mathsf{T}}$	T^{MN}	J^M	N^M
$C\,\boldsymbol{u}\wedge\boldsymbol{F}\wedge\boldsymbol{F}\wedge\boldsymbol{F}$	$6\mu^2 C\,V^{(M}B^{N)}$	$6\mu C\,B^M$	$3\frac{\mu^2}{T}C\,B^M$
$C_2\,\boldsymbol{u}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}$	$2C_2T^2V^{(M}B^{N)}$		C_2TB^M
$C_0\,\boldsymbol{u}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}\wedge\boldsymbol{F}$	$4T\mu C_0V^{(M}B^{N)}$	$2TC_0B^M$	$2\mu C_0B^M$ $-\frac{1}{2}C_0\epsilon^{MNRST}u_NA_RF_{ST}$ $+\frac{1}{3}C_0\epsilon^{MNRST}u_NA_RA_SA_T$
$C_2'\,\boldsymbol{u}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{\Omega}$	$2C_2'T^2V^{(M}\omega^{N)}$		$C_2'T\omega^M$
$C_0'\,\boldsymbol{u}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{\Omega}\wedge\boldsymbol{F}$	$2T\mu_{\mathrm{m}}C_0'V^{(M}B^{N)}$ $+2T\mu C_0'V^{(M}\omega^{N)}$ $+2TC_0'u^{(M}B^{M)}$	$TC_0'\omega^M$	$\mu C_0'\omega^M+\mu_{\mathrm{m}}C_0'B^M$ $-\frac{1}{2}C_0'\epsilon^{MNRST}u_NA_R\Omega_{ST}$

Table 5.1: One-derivative Class A and $\mathrm{H_V}$ constitutive relations for a (4+1)-dimensional null fluid. Lie algebra traces are understood in columns $\boldsymbol{\mathcal{P}}_{\mathsf{T}}$, T^{MN}, and J^M. Note that the C_0 and C_0' terms is in the anomaly polynomial are linear in $\boldsymbol{F}_{\mathsf{T}}$, hence the associated free energy current is not gauge-invariant.

Angular brackets denote a traceless symmetric combination. Note that E^M and B^M characterise the components of electric and magnetic fields in the fluid frame of reference. In lab frame, these definitions would be slightly different. We can also null reduce these definitions to obtain their respective Newton-Cartan versions

$$
\begin{aligned}
\Theta = \tilde{\mathrm{D}}_\mu u^\mu, \qquad & \sigma^{\mu\nu} = p^{\mu\rho}p^{\nu\sigma}\left(p_{\tau(\rho}\tilde{\mathrm{D}}_{\sigma)}u^\tau - \frac{1}{3}p_{\rho\sigma}\Theta\right), \\
\mathfrak{a}_\mu = u^\nu\tilde{\mathrm{D}}_\nu u_\mu, \qquad & \omega^\mu = -\epsilon^{\mu\nu\rho\sigma}n_\nu\partial_\rho u_\sigma, \\
E_\mu = F_{\mu\nu}u^\nu, \qquad & B^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}n_\nu F_{\rho\sigma}.
\end{aligned} \tag{5.8}
$$

To find the derivative corrections, let us start with the hydrostatic sector. Just like the relativistic case, Class $\mathrm{H_S}$ is empty, while there are non-trivial Class A and Class $\mathrm{H_V}$ constitutive relations. They are collectively characterised by a 7-rank thermal anomaly polynomial

$$
\begin{aligned}
\boldsymbol{\mathcal{P}}_{\mathsf{T}} = C\,\boldsymbol{u}\wedge\mathrm{tr}[\mathbf{F}\wedge\mathbf{F}\wedge\mathbf{F}] + C_2\,\boldsymbol{u}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\mathrm{tr}\,\boldsymbol{F} + C_0\,\boldsymbol{u}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\mathrm{tr}[\boldsymbol{F}\wedge\boldsymbol{F}] \\
+ C_2'\,\boldsymbol{u}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{\Omega} + C_0'\,\boldsymbol{u}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{\Omega}\wedge\mathrm{tr}\,\boldsymbol{F}.
\end{aligned} \tag{5.9}
$$

Note that we have not included a term proportional to $\boldsymbol{u}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{\Omega}\wedge\boldsymbol{\Omega}$, as it is not allowed by the adiabaticity equation; see the discussion around eq. (4.76). We have also not included the term $\boldsymbol{u}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}_{\mathsf{T}}\wedge\boldsymbol{F}_{\mathsf{T}}$, which does not contribute on torsion-less null backgrounds. Following sections 4.3.3 and 4.3.4, we can now easily read out the respective constitutive relations. The results have been summarised in table 5.1.

Next, let us consider the non-hydrostatic constitutive relations. Since the definition of "$\delta_{\mathscr{B}}$" already contains a derivative, at one-derivative order the only relevant non-hydrostatic constitutive relations are parametrised by $\mathfrak{C}_0$. It can be split into a symmetric part $\mathfrak{D}_0$ and an anti-symmetric part $\overline{\mathfrak{D}}_0$ according to eq. (2.80). Using the tensor structures we have available, we can write down 5 possible terms in $\mathfrak{D}_0$ making up Class D; they are the

$\mathfrak{D}_0$	$T^{\mu\nu}$	J^μ
$-T\zeta \begin{pmatrix} P^{MN}P^{RS} & \cdot \\ \cdot & \cdot \end{pmatrix}$	$-\zeta\, P^{MN}\Theta$	
$-T\eta \begin{pmatrix} P^{M\langle R}P^{S\rangle N} & \cdot \\ \cdot & \cdot \end{pmatrix}$	$-\eta\, \sigma^{MN}$	
$-4T\kappa \begin{pmatrix} V^{(M}P^{N)(R}V^{S)} & \cdot \\ \cdot & \cdot \end{pmatrix}$	$-2\kappa\, V^{(M}P^{N)R}\frac{1}{T}\partial_R T$	
$-2T\kappa_Q \begin{pmatrix} \cdot & V^{(M}P^{N)R} \\ V^{(R}P^{S)M} & \cdot \end{pmatrix}$	$-2\kappa_Q\, V^{(M}P^{N)R}\left(T\mathrm{D}_R\frac{\mu}{T} - E_R\right)$	$-\kappa_Q P^{MN}\frac{1}{T}\partial_N T$
$-T\sigma \begin{pmatrix} \cdot & \cdot \\ \cdot & P^{MR} \end{pmatrix}$		$-\sigma P^{MN}\left(T\mathrm{D}_N\frac{\mu}{T} - E_N\right)$

Table 5.2: One-derivative Class D constitutive relations for a $(4+1)$-dimensional null fluid.

$\overline{\mathfrak{D}}_0$	$T^{\mu\nu}$	J^μ
$-2T\overline{\kappa}_Q \begin{pmatrix} \cdot & V^{(M}P^{N)R} \\ -V^{(R}P^{S)M} & \cdot \end{pmatrix}$	$-2\overline{\kappa}_Q\, V^{(M}P^{N)R}\left(T\mathrm{D}_R\frac{\mu}{T} - E_R\right)$	$\overline{\kappa}_Q P^{MN}\frac{1}{T}\partial_N T$

Table 5.3: One-derivative Class $\overline{\mathrm{D}}$ constitutive relations for a $(4+1)$-dimensional null fluid.

bulk viscosity ζ, shear viscosity η, thermal conductivity κ, thermo-electric conductivity κ_Q, and electric conductivity σ. The results have been summarised in table 5.2. On the other hand, there is just one transport coefficient in Class $\overline{\mathrm{D}}$, denoted by $\overline{\kappa}_Q$, which is another kind of thermo-electric conductivity. The corresponding constitutive relations have been summarised in table 5.3. Finally, the quadratic form Δ associated with the Class D constitutive relations is given by

$$\begin{aligned} T\Delta = \zeta\Theta^2 + \eta\sigma^{MN}\sigma_{MN} + \sigma P^{MN}\left(T\mathrm{D}_M\frac{\mu}{T} - E_M\right)\left(T\mathrm{D}_N\frac{\mu}{T} - E_N\right) \\ + 2\kappa_Q P^{MN}\frac{1}{T}\partial_M T\left(T\mathrm{D}_N\frac{\mu}{T} - E_N\right) + \kappa P^{MN}\frac{1}{T^2}\partial_M T\partial_N T. \end{aligned} \tag{5.10}$$

Demanding it to be positive semi-definite yields a set of 4 inequality constraints among the 5 dissipative transport coefficients

$$\zeta \geq 0, \qquad \eta \geq 0, \qquad \sigma \geq 0, \qquad \kappa \geq \kappa_Q^2/\sigma. \tag{5.11}$$

These are the same as we found in section 1.1.3.

In summary, the constitutive relations of a null fluid up to the first order in derivatives are given as

$$\begin{aligned} T^{MN} &= Ru^M u^N + 2(E+P)u^{(M}V^{N)} + Pg^{MN} - \eta\,\sigma^{MN} - \zeta\, P^{MN}\Theta \\ &\quad - 2V^{(M}P^{N)R}\left(\kappa\frac{1}{T}\partial_R T + (\kappa_Q + \overline{\kappa}_Q)\left(T\mathrm{D}_R\frac{\mu}{T} - E_R\right)\right) \\ &\quad + 2\left(3\mu^2 C + C_2 T^2 + 2T\mu C_0 + T\mu_{\mathrm{m}} C_0'\right)V^{(M}B^{N)} + 2\left(C_2' T^2 + T\mu C_0'\right)V^{(M}\omega^{N)} \\ &\quad + 2TC_0' u^{(M}B^{N)} + \mathcal{O}(\partial^2), \\ J^M &= Qu^M - (\kappa_Q - \overline{\kappa}_Q)\, P^{MN}\frac{1}{T}\partial_N T - \sigma\, P^{MN}\left(T\mathrm{D}_N\frac{\mu}{T} - E_N\right) \\ &\quad + (6\mu C + 2TC_0)\, B^M + TC_0'\omega^M + \mathcal{O}(\partial^2). \end{aligned} \tag{5.12}$$

The associated free energy and entropy currents are

$$
\begin{aligned}
N^M &= \frac{1}{T} P u^M + \left(3\frac{\mu^2}{T}C + C_2 T + 2\mu C_0 + \mu_{\mathrm{m}} C_0'\right) B^M + \left(TC_2' + \mu C_0'\right)\omega^M \\
&\quad - C_0 \epsilon^{MNRST} u_N \left(\frac{1}{2}A_R F_{ST} - \frac{1}{3}A_R A_S A_T\right) - \frac{1}{2}C_0' \epsilon^{MNRST} u_N A_R \Omega_{ST} + \mathcal{O}(\partial^2), \\
J_{\mathrm{S}}^M &= S u^M + \left(\frac{\mu_{\mathrm{m}}}{T}\kappa + \frac{\mu}{T}\left(\kappa_Q - \overline{\kappa}_Q\right)\right) P^{MN} \frac{1}{T}\partial_N T \\
&\quad + \left(\frac{\mu}{T}\sigma + \frac{\mu_{\mathrm{m}}}{T}\left(\kappa_Q + \overline{\kappa}_Q\right)\right) P^{MN}\left(T\mathrm{D}_N \frac{\mu}{T} - E_N\right) \\
&\quad + \left(2C_2 T + 2\mu C_0 + \mu_{\mathrm{m}} C_0'\right) B^M + \left(2TC_2' + \mu C_0'\right)\omega^M \\
&\quad - C_0 \epsilon^{MNRST} u_N \left(\frac{1}{2}A_R F_{ST} - \frac{1}{3}A_R A_S A_T\right) - \frac{1}{2}C_0' \epsilon^{MNRST} u_N A_R \Omega_{ST} + \mathcal{O}(\partial^2). \qquad (5.13)
\end{aligned}
$$

They satisfy the second law of thermodynamics, provided that the ideal order transport coefficients satisfy the thermodynamic relations and the first order dissipative transport coefficients satisfy their respective inequalities. Finally, performing the null reduction yields us the Galilean fluid constitutive relations

$$
\begin{aligned}
\rho^\mu &= R u^\mu + TC_0' B^\mu + \mathcal{O}(\partial^2), \\
\epsilon^\mu &= \left(E + \frac{1}{2} R u^a u_a + TC_0' B^\nu \bar{u}_\nu\right) u^\mu + P\bar{u}^\mu - \eta\, \sigma^{\mu\nu}\bar{u}_\nu - \zeta\, \Theta \bar{u}^\mu \\
&\quad - h^{\mu\nu}\left(\kappa \frac{1}{T}\partial_\nu T + \left(\kappa_Q + \overline{\kappa}_Q\right)\left(T\mathrm{D}_\nu \frac{\mu}{T} - E_\nu\right)\right) + \left(C_2' T^2 + T\mu C_0'\right)\omega^\mu \\
&\quad + \left(3\mu^2 C + C_2 T^2 + 2T\mu C_0 + T\mu_{\mathrm{m}} C_0' + \frac{1}{2} TC_0' u^a u_a\right) B^\mu + \mathcal{O}(\partial^2) \\
p^\mu{}_a &= R u^\mu u_a + P e^\mu{}_a - \eta\, \sigma^{\mu\nu} e_{\nu a} - \zeta\, \Theta e^\mu{}_a + TC_0' \left(u^\mu B_a + B^\mu u_a\right) + \mathcal{O}(\partial^2), \\
j^\mu &= Q u^\mu - \left(\kappa_Q - \overline{\kappa}_Q\right) h^{\mu\nu} \frac{1}{T}\partial_\nu T - \sigma\, h^{\mu\nu}\left(T\mathrm{D}_\nu \frac{\mu}{T} - E_\nu\right) \\
&\quad + \left(6\mu C + 2TC_0\right) B^\mu + TC_0' \omega^\mu + \mathcal{O}(\partial^2). \qquad (5.14)
\end{aligned}
$$

If we instead wanted these results in the mass frame, we could perform a frame transformation $u^\mu \to u^\mu - TC_0' B^\mu / R$. In this frame, the mass current is just $\rho^\mu_{\mathrm{mass}} = R u^\mu$, while the other currents look like

$$
\begin{aligned}
\epsilon^\mu_{\mathrm{mass}} &= \left(E + \frac{1}{2} R u^a u_a\right) u^\mu + P\bar{u}^\mu - h^{\mu\nu}\left(\kappa \frac{1}{T}\partial_\nu T + \left(\kappa_Q + \overline{\kappa}_Q\right)\left(T\mathrm{D}_\nu \frac{\mu}{T} - E_\nu\right)\right) \\
&\quad + \xi_B' B^\mu + \xi_\omega' \omega^\mu - \eta\, \sigma^{\mu\nu}\bar{u}_\nu - \zeta\, \Theta \bar{u}^\mu + \mathcal{O}(\partial^2), \\
p_{\mathrm{mass}}{}^\mu{}_a &= R u^\mu u_a + P e^\mu{}_a - \eta\, \sigma^{\mu\nu} e_{\nu a} - \zeta\, \Theta e^\mu{}_a + \mathcal{O}(\partial^2), \\
j^\mu_{\mathrm{mass}} &= Q u^\mu - \left(\kappa_Q - \overline{\kappa}_Q\right) h^{\mu\nu} \frac{1}{T}\partial_\nu T - \sigma\, h^{\mu\nu}\left(T\mathrm{D}_\nu \frac{\mu}{T} - E_\nu\right) \\
&\quad + \xi_B B^\mu + \xi_\omega \omega^\mu + \mathcal{O}(\partial^2). \qquad (5.15)
\end{aligned}
$$

Here we have defined the parity-odd transport coefficients

$$\xi_B = 6\mu C + 2TC_0 - \frac{Q}{R}TC_0', \qquad \xi_\omega = TC_0',$$
$$\xi_B' = 3\mu^2 C + C_2 T^2 + 2T\mu C_0 - \frac{E+P-\mu_{\rm m}R}{R}TC_0', \qquad \xi_\omega' = C_2' T^2 + T\mu C_0'. \tag{5.16}$$

This is the standard form of the first order Galilean hydrodynamics, which was first derived in full generality in [3]. Note that these constitutive relations still satisfy $n_\mu p^\mu{}_a = \rho^\mu e_{a\mu} = R u_a$ and $p^{\mu[a}e^{b]}{}_\mu = 0$, so the boost and rotation Ward identities are identically satisfied.

5.2 Galilean superfluids

In this example, we study Abelian Galilean superfluids. These are essentially Galilean fluids with a spontaneously broken U(1) symmetry. The associated Goldstone phase field φ serves as a gapless mode in the hydrodynamic description. The first theory of Galilean superfluid dynamics was written down by London [129] and was later elaborated upon by Landau and Tisza [130, 131], to describe the phenomenology of liquid ^{2}He. The ideal order results can be found in [10], which we have extended to include one-derivative corrections in our work [4]. The results presented here have been taken directly from [4].

5.2.1 Goldstone modes and Josephson equation

Under an infinitesimal symmetry transformation parametrised by $\mathcal{X} = (\chi^M, \Lambda_\chi)$, the Goldstone mode φ transforms as $\delta_\mathcal{X}\varphi = \chi^M\partial_M\varphi - \Lambda_\chi$. We can define a covariant superfluid velocity by a taking a derivative of φ, leading to

$$\xi_M = \partial_M\varphi + A_M, \tag{5.17}$$

which satisfies $2\partial_{[M}\xi_{N]} = F_{MN}$. We demand $\delta_\mathcal{V}\varphi = V^M\partial_M\varphi - \Lambda_V = -1$. The value -1 is purely a choice; different choices yield the same results up to a hydrodynamic frame transformation. With the current choice, we can make a direct comparison of our results with Landau [10]. Due to the compatibility condition of the connection A_M, it follows that

$$V^M A_M + \Lambda_V = 0 \implies V^M\xi_M = -1. \tag{5.18}$$

Let us call the equation of motion for φ to be

$$K \approx 0. \tag{5.19}$$

Following eq. (4.58), we can write down an adiabaticity equation for null superfluids

$$\mathrm{D}_M N^M = \frac{1}{2}T^{MN}\delta_\mathcal{B} g_{MN} + J^M\delta_\mathcal{B}A_M + K\delta_\mathcal{B}\varphi + \Delta, \qquad \Delta \geq 0. \tag{5.20}$$

where

$$\delta_{\mathscr{B}}\varphi = \beta^M \partial_M \varphi - \Lambda_\beta = \frac{1}{T}\left(u^M \xi_M + \mu_{\mathrm{m}} - \mu\right). \tag{5.21}$$

Recycling our relativistic discussion from section 3.2.1, we can solve the adiabaticity equation (5.20) at the zeroth order in derivatives and get the Josephson equation for null superfluids

$$K = -\alpha \delta_{\mathscr{B}}\varphi \approx 0 \qquad \Longrightarrow \qquad u^M \xi_M \approx \mu - \mu_{\mathrm{m}} + \mathcal{O}(\partial). \tag{5.22}$$

Note that this equation is slightly different compared to relativistic superfluids, due to the presence of the mass chemical potential μ_{m} on the right hand side. This is due to the fact that in a Galilean superfluid, a linear combination of U(1)-mass and U(1)-internal symmetry is broken, while the other linear combination remains unbroken. In the null reduced form, it equivalently implies

$$-\mu_s - \frac{1}{2} h_{\mu\nu} \zeta^\mu \zeta^\nu \approx \mu - \mu_{\mathrm{m}} + \mathcal{O}(\partial), \tag{5.23}$$

where $\zeta^\mu = \xi^\mu - u^\mu$ is the relative velocity of the superfluid component with respect to the ordinary fluid component. The scalar $\mu_s = -\frac{1}{2}\xi^M \xi_M$, on the other hand, is called the superfluid potential. In practise, the Josephson equation can be used to eliminate one of the chemical potentials. In [10] e.g., Landau is working in a setup with no U(1) chemical potential, i.e. $\mu = 0$, and has implicitly eliminated the mass chemical potential via $\mu_{\mathrm{m}} \approx \mu_s + \frac{1}{2} h_{\mu\nu}\zeta^\mu \zeta^\nu$. Care should be taken when comparing the results, because what we are calling μ_s is denoted by μ in [10].

5.2.2 Ideal null superfluids

Let us now work out the ideal null superfluids. As always, Class $\mathrm{H_S}$ is the only non-empty class for ideal null superfluids as well. However, the hydrostatic scalar density $\mathcal{N}$ characterising Class $\mathrm{H_S}$ is given by a function of four variables $P(T, \mu, \mu_{\mathrm{m}}, \mu_s)$. We have omitted the only other possible ideal order scalar $u^M \xi_M$ in the functional dependence, because it is not independent on-shell due to the Josephson equation. We have already provided the $\delta_{\mathscr{B}}$ variations of T, μ_{m}, and μ in eq. (5.1); for μ_s we find

$$\delta_{\mathscr{B}}\mu_s = \frac{1}{2}\xi^M \xi^N \delta_{\mathscr{B}} g_{\mu\nu} - \xi^M \delta_{\mathscr{B}} A_M - \xi^M \mathrm{D}_M \delta_{\mathscr{B}}\varphi. \tag{5.24}$$

We can now compute the divergence of the free energy current

$$\begin{aligned}
\mathrm{D}_M\left(\beta^M P\right) &= \frac{1}{\sqrt{-g}}\delta_{\mathscr{B}}\left(\sqrt{-g}P\right) \\
&= \frac{1}{2} P g^{\mu\nu}\delta_{\mathscr{B}} g_{\mu\nu} + \frac{\partial P}{\partial T}\delta_{\mathscr{B}} T + \frac{\partial P}{\partial \mu}\delta_{\mathscr{B}}\mu + \frac{\partial P}{\partial \mu_{\mathrm{m}}}\delta_{\mathscr{B}}\mu_{\mathrm{m}} + \frac{\partial P}{\partial \mu_s}\delta_{\mathscr{B}}\mu_s, \\
&= \left(R u^M u^N + 2E u^{(M}V^{N)} + P P^{MN} + R_s \xi^M \xi^N\right)\frac{1}{2}\delta_{\mathscr{B}} g_{MN} \\
&\quad + \left(Q u^M - R_s \xi^N\right)\delta_{\mathscr{B}} A_M + \mathrm{D}_M\left(R_s \xi^M\right)\delta_{\mathscr{B}}\varphi - \mathrm{D}_M\left(R_s \xi^M \delta_{\mathscr{B}}\varphi\right),
\end{aligned} \tag{5.25}$$

where we have defined

$$S = \frac{\partial P}{\partial T}, \qquad R = \frac{\partial P}{\partial \mu_{\rm m}}, \qquad Q = \frac{\partial P}{\partial \mu}, \qquad R_s = \frac{\partial P}{\partial \mu_s},$$
$$E = T\frac{\partial P}{\partial T} + \mu_{\rm m}\frac{\partial P}{\partial \mu_{\rm m}} + \mu\frac{\partial P}{\partial \mu} - P. \tag{5.26}$$

Comparing eq. (5.25) with eq. (2.74), we can read out the ideal null superfluid constitutive relations, free energy, and entropy currents

$$\begin{aligned}
T^{MN} &= Ru^M u^N + 2(E+P)u^{(M}V^{N)} + Pg^{MN} + R_s\xi^M\xi^N + \mathcal{O}(\partial), \\
J^M &= Qu^M - R_s\xi^M + \mathcal{O}(\partial), \\
K &= -\alpha\delta_{\mathscr{B}}\varphi + \mathrm{D}_M(R_s\xi^M) + \mathcal{O}(\partial), \\
N^M &= \frac{1}{T}Pu^M + \delta_{\mathscr{B}}\varphi R_s\xi^M + \mathcal{O}(\partial), \\
J^M_{\rm S} &= N^M - \left(T^{MN}\beta_N + \frac{\mu}{T}J^M\right) = Su^M + \mathcal{O}(\partial).
\end{aligned} \tag{5.27}$$

In the entropy current, we have ignored a term proportional to V^M, as it does not contribute to the second law. Due to the definitions (5.26), the coefficients appearing above satisfy the thermodynamic relations

$$\begin{aligned}
\text{Gibbs-Duhem equation:} &\qquad \mathrm{d}P = S\mathrm{d}T + Q\mathrm{d}\mu + R\mathrm{d}\mu_{\rm m} + R_s\mathrm{d}\mu_s, \\
\text{Euler scaling relation:} &\qquad E + P = ST + Q\mu, \\
\text{First law of thermodynamics:} &\qquad \mathrm{d}E = T\mathrm{d}S + \mu\mathrm{d}Q + \mu_{\rm m}\mathrm{d}R - R_s\mathrm{d}\mu_s.
\end{aligned} \tag{5.28}$$

Using eq. (5.27), we can work out the first correction to the Josephson equation, leading to

$$u^M\xi_M = \mu - \mu_{\rm m} + \frac{T}{\alpha}\mathrm{D}_M\left(R_s\xi^M\right) + \mathcal{O}(\partial). \tag{5.29}$$

We can perform the null reduction on the ideal null superfluid constitutive relations to obtain their respective Galilean form

$$\begin{aligned}
\rho^\mu &= Ru^\mu + R_s\xi^\mu + \mathcal{O}(\partial), \\
\epsilon^\mu &= \left(E + \frac{1}{2}Ru^a u_a\right)u^\mu + R_s\xi^\mu\left(\mu_s + \frac{1}{2}\xi^a\xi_a\right) + P\bar{u}^\mu + \mathcal{O}(\partial) \\
p^\mu{}_a &= Ru^\mu u_a + Pe_a{}^\mu + R_s\xi^\mu\xi_a + \mathcal{O}(\partial), \qquad j^\mu = Qu^\mu - R_s\xi^\mu + \mathcal{O}(\partial), \\
n^\mu &= \frac{1}{T}Pu^\mu + \mathcal{O}(\partial), \qquad s^\mu = Su^\mu + \mathcal{O}(\partial).
\end{aligned} \tag{5.30}$$

Like ordinary Galilean fluids, P is the isotropic pressure of the superfluid, while R, E, Q, and S are its mass, energy, charge, and entropy densities respectively. The coefficient R_s can be identified with the superfluid density. Note that the linear combination $\rho^\mu + j^\mu$ does not have any R_s dependence, because the associated linear combination of U(1)-mass and U(1)-internal symmetry is unbroken. The Josephson equation, on the other hand, becomes

$$-\mu_s - \frac{1}{2}h_{\mu\nu}\zeta^\mu\zeta^\nu = \mu - \mu_{\rm m} + \frac{T}{\alpha}\tilde{\mathrm{D}}_\mu\left(R_s\xi^\mu\right) + \mathcal{O}(\partial). \tag{5.31}$$

In the limit $\mu = 0$, these results can be compared with their textbook version in [10].

5.2.3 One-derivative corrections

Let us talk about one-derivative corrections to these constitutive relations. In table 5.4, we have provided a list of one-derivative tensor structures that we require in the following discussion. Note that this list is not complete; it only contains terms that show up in our calculation. We have defined $\zeta^M = P^{MN}\xi_N$ and $P_\zeta^{MN} = P^{MN} - \zeta^M\zeta^N/\zeta^2$, along with their null reduced form $\zeta^\mu = \xi^\mu - u^\mu$ and $p_\zeta^{\mu\nu} = p^{\mu\nu} - \zeta^\mu\zeta^\nu/\zeta^2$.

Beginning with the hydrostatic sector, we note that the Class $\mathrm{H_S}$ constitutive relations are now characterised by a hydrostatic scalar density $\mathcal{N}$ given as

$$\mathcal{N} = P + f_1\, S_1 + f_2\, S_2 + f_3\, S_3 + g_1\, \tilde{S}_{e,1} + g_2\, \tilde{S}_{e,2} + g_3\, \tilde{S}_{e,3}. \tag{5.32}$$

We have chosen not to include the first order hydrostatic scalars: $\mathrm{D}_M\xi^M$ and $\xi^M\partial_M\mu_s$, as the former is a total derivative and the latter can be exchanged for $\mathrm{D}_M(R_s\xi^M)$, which is not hydrostatic due to the Josephson equation. We define the derivatives of the transport coefficients appearing above as

$$\begin{aligned}
\mathrm{d}f_i &= \frac{1}{T}\alpha_{E,i}\mathrm{d}T + T\alpha_{R_n,i}\mathrm{d}\frac{\mu_\mathrm{m}}{T} + T\alpha_{Q,i}\mathrm{d}\frac{\mu}{T} - \frac{1}{2}\left(\alpha_{R_s,i} - \frac{f_i}{2\hat{\mu}_s}\right)\mathrm{d}\zeta^2, \\
\mathrm{d}g_i &= \frac{1}{T}\tilde{\alpha}_{E,i}\mathrm{d}T + T\tilde{\alpha}_{R_n,i}\mathrm{d}\frac{\mu_\mathrm{m}}{T} + T\tilde{\alpha}_{Q,i}\mathrm{d}\frac{\mu}{T} - \frac{1}{2}\left(\tilde{\alpha}_{R_s,i} - \frac{g_i}{2\hat{\mu}_s}\right)\mathrm{d}\zeta^2,
\end{aligned} \tag{5.33}$$

along with

$$\alpha_{E,i} + f_i = \alpha_{S,i}T + \alpha_{Q,i}\mu + \alpha_{R_n,i}\mu_\mathrm{m}, \qquad \tilde{\alpha}_{E,i} + g_i = \tilde{\alpha}_{S,i}T + \tilde{\alpha}_{Q,i}\mu + \tilde{\alpha}_{R_n,i}\mu_\mathrm{m}. \tag{5.34}$$

By performing a $\delta_\mathscr{B}$ variation of $\mathcal{N}$ and comparing it to eq. (2.74), we can read out the respective Class $\mathrm{H_S}$ constitutive relations. The results have been summarised in table 5.5. Now for Class A and Class $\mathrm{H_V}$, we can directly import the results from the respective ordinary fluid calculation in table 5.1. We should note, however, that due to the presence of a "gauge fixed" version of the gauge field $\xi_M = A_M + \partial_M\varphi$, three constants in Class $\mathrm{H_V}$, i.e. C_2, C_0, and C_0', can be removed by shifting g_1, g_2, and g_3. Therefore, only the C_2' term in Class $\mathrm{H_V}$ and the C term in Class A remain independent for a Galilean superfluid.

Finally, let us consider the non-hydrostatic sector. We need to write the most generic expression for $\mathfrak{C}_0$, which decomposes into a symmetric part $\mathfrak{D}_0$ and an antisymmetric part $\overline{\mathfrak{D}}_0$. They correspond to Class D and Class $\overline{\mathrm{D}}$ constitutive relations respectively. There are a total of 20 transport coefficients in Class $\overline{\mathrm{D}}$: $[\beta_{[ij]}]_{5\times5}$, $[\kappa_{[ij]}]_{3\times3}$, $[\tilde{\kappa}_{(ij)}]_{3\times3}$, and $\tilde{\eta}$. On the other hand, there are 25 transport coefficients in Class D: $[\beta_{(ij)}]_{5\times5}$, $[\kappa_{(ij)}]_{3\times3}$, $[\tilde{\kappa}_{[ij]}]_{3\times3}$, and η, including $\beta_{55} = \alpha/T$ discussed in section 5.2.1. The results have been summarised in tables 5.6 and 5.7. The associated quadratic form Δ takes the form

$$T\Delta = \sum_{i,j=1}^{5} S_i\beta_{(ij)}S_j + \left(\sum_{i,j=1}^{3} V_i^M\kappa_{(ij)}V_{j,M} + \sum_{i,j=1}^{3} V_i^M\tilde{\kappa}_{[ij]}\tilde{V}_{j,M}\right) + \eta\sigma^{MN}\sigma_{MN}. \tag{5.35}$$

		Null background	Newton-Cartan background
Non-hydrostatic — on-shell independent			
S_1	$\frac{T}{2}P_\zeta^{MN}\delta_{\mathscr{B}}g_{MN}$	$P_\zeta^{MN}\mathrm{D}_M u_N$	$p_\zeta{}^\mu{}_\nu\tilde{\mathrm{D}}_\mu u^\nu$
S_2	$TV^M\zeta^N\delta_{\mathscr{B}}g_{MN}$	$\zeta^M\left(\frac{1}{T}\partial_M T + u^N H_{NM}\right)$	$\zeta^\mu\left(\frac{1}{T}\partial_\mu T + u^\nu H_{\nu\mu}\right)$
S_3	$\frac{T}{2}\zeta^M\zeta^N\delta_{\mathscr{B}}g_{MN}$	$\zeta^M\zeta^N\mathrm{D}_M u_N$	$\zeta^\mu\zeta_\nu\tilde{\mathrm{D}}_\mu u^\nu$
S_4	$T\zeta^M\delta_{\mathscr{B}}A_M$	$\zeta^M\left(T\partial_M\frac{\mu}{T} - E_M\right)$	$\zeta^\mu\left(T\partial_\mu\frac{\mu}{T} - E_\mu\right)$
S_5	$T\delta_{\mathscr{B}}\varphi$	$u^M\xi_M + \mu_{\mathrm{m}} - \mu$	$-\frac{1}{2}\zeta^a\zeta_a - \mu_s + \mu_{\mathrm{m}} - \mu$
V_1^M	$TP_\zeta^{MR}V^N\delta_{\mathscr{B}}g_{RN}$	$P_\zeta^{MN}\left(\frac{1}{T}\partial_N T + u^R H_{RN}\right)$	$p_\zeta^{\mu\nu}\left(\frac{1}{T}\partial_\nu T + u^\rho H_{\rho\nu}\right)$
V_2^M	$TP_\zeta^{MR}\zeta^N\delta_{\mathscr{B}}g_{RN}$	$2P_\zeta^{MR}\zeta^N\mathrm{D}_{(R}u_{N)}$	$2p_\zeta^{\mu\nu}\zeta^\sigma p_{\rho(\sigma}\tilde{\mathrm{D}}_{\nu)}u^\rho$
V_3^M	$TP_\zeta^{MN}\delta_{\mathscr{B}}A_N$	$P_\zeta^{MN}\left(T\partial_N\frac{\mu}{T} - E_N\right)$	$p_\zeta^{\mu\nu}\left(T\partial_\nu\frac{\mu}{T} - E_\nu\right)$
σ_ζ^{MN}	$\frac{T}{2}P_\zeta^{R\langle M}P_\zeta^{N\rangle S}\delta_{\mathscr{B}}g_{RS}$	$P_\zeta^{MR}P_\zeta^{NS}\left(\mathrm{D}_{(R}u_{S)} - \frac{1}{2}P^\zeta_{RS}S_1\right)$	$p_\zeta^{\mu\rho}p_\zeta^{\nu\sigma}\left(p_{\tau(\rho}\tilde{\mathrm{D}}_{\sigma)}u^\tau - \frac{p^\zeta_{\rho\sigma}}{2}S_1\right)$
$\tilde{V}_1^M$		$\epsilon^{MNRST}V_N u_R\zeta_S V_{1,T}$	$-\epsilon^{\mu\nu\rho\sigma}n_\nu\zeta_\rho V_{1,\sigma}$
$\tilde{V}_2^M$		$\epsilon^{MNRST}V_N u_R\zeta_S V_{2,T}$	$-\epsilon^{\mu\nu\rho\sigma}n_\nu\zeta_\rho V_{2,\sigma}$
$\tilde{V}_3^M$		$\epsilon^{MNRST}V_N u_R\zeta_S V_{3,T}$	$-\epsilon^{\mu\nu\rho\sigma}n_\nu\zeta_\rho V_{3,\sigma}$
$\tilde{\sigma}_\zeta^{MN}$		$P_\zeta^{L(M}\epsilon^{N)RSTP}V_R u_S\zeta_T\sigma^\zeta_{PL}$	$-p_\zeta^{\lambda(\mu}\epsilon^{\mu)\rho\sigma\tau}n_\rho\zeta_\sigma\sigma_{\tau\lambda}$
Non-hydrostatic — on-shell dependent			
S_6	$Tu^M V^N\delta_{\mathscr{B}}g_{MN}$	$\frac{1}{T}u^M\partial_M T$	$\frac{1}{T}u^\mu\partial_\mu T$
S_7	$Tu^M\delta_{\mathscr{B}}A_M$	$Tu^M\partial_M\frac{\mu}{T}$	$Tu^\mu\partial_\mu\frac{\mu}{T}$
S_8	$\frac{T}{2}u^M u^N\delta_{\mathscr{B}}g_{MN}$	$Tu^M\partial_M\frac{\mu_{\mathrm{m}}}{T}$	$Tu^\mu\partial_\mu\frac{\mu_{\mathrm{m}}}{T}$
S_9	$Tu^M\zeta^N\delta_{\mathscr{B}}g_{MN}$	$\zeta^M\left(T\partial_M\frac{\mu_{\mathrm{m}}}{T} + \mathfrak{a}_M\right)$	$\zeta^\mu\left(T\partial_\mu\frac{\mu_{\mathrm{m}}}{T} + \mathfrak{a}_\mu\right)$
V_4^M	$TP_\zeta^{MR}u^N\delta_{\mathscr{B}}g_{RN}$	$P_\zeta^{MN}\left(T\partial_N\frac{\mu_{\mathrm{m}}}{T} + \mathfrak{a}_N\right)$	$p_\zeta^{\mu\nu}\left(T\partial_\nu\frac{\mu_{\mathrm{m}}}{T} + \mathfrak{a}_\nu\right)$
$\tilde{V}_4^M$		$\epsilon^{MNRST}V_N u_R\zeta_S V_{4,T}$	$-\epsilon^{\mu\nu\rho\sigma}n_\nu\zeta_\rho V_{4,\sigma}$
Hydrostatic			
$S_{e,1}$		$\frac{1}{T}\zeta^M\partial_M T$	$\frac{1}{T}\zeta^\mu\partial_\mu T$
$S_{e,2}$		$T\zeta^M\partial_M\frac{\mu}{T}$	$T\zeta^\mu\partial_\mu\frac{\mu}{T}$
$S_{e,3}$		$T\zeta^M\partial_M\frac{\mu_{\mathrm{m}}}{T}$	$T\zeta^\mu\partial_\mu\frac{\mu_{\mathrm{m}}}{T}$
$V_{e,1}^M$		$\frac{1}{T}P_\zeta^{MN}\partial_N T$	$\frac{1}{T}p_\zeta^{\mu\nu}\partial_\nu T$
$V_{e,2}^M$		$TP_\zeta^{MN}\partial_N\frac{\mu}{T}$	$Tp_\zeta^{\mu\nu}\partial_\nu\frac{\mu}{T}$
$V_{e,3}^M$		$TP_\zeta^{MN}\partial_N\frac{\mu_{\mathrm{m}}}{T}$	$Tp_\zeta^{\mu\nu}\partial_\nu\frac{\mu_{\mathrm{m}}}{T}$
$\tilde{S}_{e,1}$		$T\epsilon^{MNRST}\zeta_M V_N u_R\partial_S u_T$	$T\epsilon^{\mu\nu\rho\sigma}n_\mu\zeta_\nu\partial_\rho u_\sigma$
$\tilde{S}_{e,2}$		$\frac{1}{2}T\epsilon^{MNRST}\zeta_M V_N u_R F_{ST}$	$\frac{T}{2}\epsilon^{\mu\nu\rho\sigma}n_\mu\zeta_\nu F_{\rho\sigma}$
$\tilde{S}_{e,3}$		$\frac{1}{2}T\epsilon^{MNRST}\zeta_M V_N u_R H_{ST}$	$\frac{T}{2}\epsilon^{\mu\nu\rho\sigma}n_\mu\zeta_\nu H_{\rho\sigma}$
$\tilde{V}_{e,1}^M$		$TP_\zeta{}^M{}_K\epsilon^{KNRST}V_N u_R\partial_S u_T$	$-Tp_\zeta{}^\mu{}_\tau\varepsilon^{\tau\nu\rho\sigma}n_\nu\partial_\rho u_\sigma$
$\tilde{V}_{e,2}^M$		$\frac{1}{2}TP_\zeta{}^M{}_K\epsilon^{KNRST}V_N u_R F_{ST}$	$-\frac{T}{2}p_\zeta{}^\mu{}_\tau\varepsilon^{\tau\nu\rho\sigma}n_\nu F_{\rho\sigma}$
$\tilde{V}_{e,3}^M$		$TP_\zeta{}^M{}_K\epsilon^{KNRST}\xi_N u_R\partial_S u_T$	$Tp_\zeta{}^\mu{}_\tau\varepsilon^{\tau\nu\rho\sigma}\zeta_\nu\partial_\rho u_\sigma$ $+(\mu_s + \frac{1}{2}\zeta^\mu\zeta_\mu)\tilde{V}^\mu_{e,1}$
$\tilde{V}_{e,4}^M$		$\frac{1}{2}TP_\zeta{}^M{}_K\epsilon^{KNRST}\xi_N u_R F_{ST}$	$\frac{T}{2}p_\zeta{}^\mu{}_\tau\varepsilon^{\tau\nu\rho\sigma}\zeta_\nu F_{\rho\sigma}$ $+(\mu_s + \frac{1}{2}\zeta^\mu\zeta_\mu)\tilde{V}^\mu_{e,2}$
$\vdots$		$\vdots$	$\vdots$

Table 5.4: First order data for 5-dimensional null and 4-dimensional Galilean superfluids. Note that for null fluids coupled to torsionless backgrounds, we must switch off H_{MN}, setting $S_2 = S_{e,1}$, $V_1^M = V_{e,1}^M$, and $\tilde{S}_{e,3} = 0$. However, in the intermediate steps, they are independent.

$\mathcal{N}$	T^{MN}	J^M
$f_1\, S_{e,1}$	$\left(\alpha_{R_n,1} u^M u^N + 2\alpha_{E,1} V^{(M} u^{N)} + \alpha_{R_s,1}\lambda^{MN} + f_1 P_\zeta^{MN}\right) S_{e,1}$ $-2 f_1 \xi^{(M} V_{e,1}^{N)} + 2 f_1 \zeta^{(M} V^{N)} S_6 - 2 V^{(M} u^{N)} \frac{1}{T} \mathrm{D}_R (T f_1 \zeta^R)$	$(\alpha_{Q,1} u^M - \alpha_{R_s,1}\zeta^M)\, S_{e,1}$ $+ f_1 V_{e,1}^M$
$f_2\, S_{e,2}$	$\left(\alpha_{R_n,2} u^M u^N + 2\alpha_{E,2} V^{(M} u^{N)} + \alpha_{R_s,2}\lambda^{MN} + f_2 P_\zeta^{MN}\right) S_{e,2}$ $-2 f_2 \xi^{(M} V_{e,2}^{N)} + 2 f_2 \zeta^{(M} V^{N)} S_7$	$(\alpha_{Q,2} u^M - \alpha_{R_s,2}\zeta^M)\, S_{e,2}$ $+ f_2 V_{e,2}^M$ $- u^M \frac{1}{T} \mathrm{D}_R (T f_2 \zeta^R)$
$f_3\, S_{e,3}$	$\left(\alpha_{R_n,3} u^M u^N + 2\alpha_{E,3} V^{(M} u^{N)} + \alpha_{R_s,3}\lambda^{MN} + f_3 P_\zeta^{MN}\right) S_{e,3}$ $-2 f_3 \xi^{(M} V_{e,3}^{N)} + 2 f_3 \zeta^{(M} V^{N)} S_8 - u^M u^N \frac{1}{T} \mathrm{D}_R (T f_3 \zeta^R)$	$(\alpha_{Q,3} u^M - \alpha_{R_s,3}\zeta^M)\, S_{e,3}$ $+ f_3 V_{e,3}^M$
$g_1\, \tilde{S}_{e,1}$	$\left(u^M u^N \tilde{\alpha}_{R_n,1} + 2 V^{(M} u^{N)} \tilde{\alpha}_{E,1} + \tilde{\alpha}_{R_s,1}\lambda^{MN} + g_1 \frac{\zeta^M \zeta^N}{\zeta^2}\right) \tilde{S}_{e,1}$ $+2 u^{(M} V^{N)} g_1 S_{e,1} + 2 u^{(M} \epsilon^{N)KRST} \mathrm{D}_K \left(T g_1 V_R u_S \xi_T\right)$ $-2 g_1 V^{(M} \tilde{V}_{e,3}^{N)} - 2 g_1 u^{(M} \tilde{V}_{e,1}^{N)}$	$(\tilde{\alpha}_{Q,1} u^M - \tilde{\alpha}_{R_s,1}\zeta^M)\, \tilde{S}_{e,1}$ $+ g_1 \tilde{V}_{e,1}^M$
$g_2\, \tilde{S}_{e,2}$	$\left(u^M u^N \tilde{\alpha}_{R_n,2} + 2 V^{(M} u^{N)} \tilde{\alpha}_{E,2} + \tilde{\alpha}_{R_s,2}\lambda^{MN} + g_2 \frac{\zeta^M \zeta^N}{\zeta^2}\right) \tilde{S}_{e,2}$ $-2 g_4 V^{(M} \tilde{V}_{e,4}^{N)} - 2 g_2 u^{(M} \tilde{V}_{e,2}^{N)}$	$(\tilde{\alpha}_{Q,2} u^M - \tilde{\alpha}_{R_s,2}\zeta^M)\, \tilde{S}_{e,2}$ $+ g_2 \tilde{V}_{e,2}^M +$ $\epsilon^{MNRST} \mathrm{D}_N \left(T g_2 V_R u_S \xi_T\right)$
$g_3\, \tilde{S}_{e,3}$	$2 V^{(M} \epsilon^{N)KRST} \mathrm{D}_K \left(g_3 T V_R u_S \zeta_T\right)$	

$\mathcal{N}$	K	N^M
$f_1\, S_{e,1}$	$\mathrm{D}_M \left(\zeta^M \alpha_{R_s,1} S_{e,1} - f_1 V_{e,1}^M\right)$	$\frac{1}{T} f_1 \left(u^\mu S_{e,1} - \zeta^\mu S_6\right)$
$f_2\, S_{e,2}$	$\mathrm{D}_M \left(\zeta^M \alpha_{R_s,2} S_{e,2} - f_1 V_{e,2}^M\right)$	$\frac{1}{T} f_2 \left(u^\mu S_{e,2} - \zeta^\mu S_7\right)$
$f_3\, S_{e,3}$	$\mathrm{D}_M \left(\zeta^M \alpha_{R_s,3} S_{e,3} - f_3 V_{e,3}^M\right)$	$\frac{1}{T} f_3 \left(u^\mu S_{e,3} - \zeta^\mu S_8\right)$
$g_1\, \tilde{S}_{e,1}$	$\mathrm{D}_M \left(\zeta^M \tilde{\alpha}_{R_s,1} \tilde{S}_{e,1} - g_1 \tilde{V}_{e,1}^M\right)$	$g_1 \left(\frac{1}{T} u^M \tilde{S}_{e,1} + \tilde{V}_4^M\right)$
$g_2\, \tilde{S}_{e,2}$	$\mathrm{D}_M \left(\zeta^M \tilde{\alpha}_{R_s,2} \tilde{S}_{e,2} - g_2 \tilde{V}_{e,2}^M\right)$	$g_2 \left(\frac{1}{T} u^M \tilde{S}_{e,2} + \tilde{V}_3^M\right)$
$g_3\, \tilde{S}_{e,3}$		$g_3\, \tilde{V}_1^M$

Table 5.5: One-derivative Class $\mathrm{H_S}$ constitutive relations for a $(3+1)$-dimensional relativistic superfluid. We have defined $\lambda^{MN} = \zeta^M \zeta^N + 2\zeta^{(M} u^{N)} - 2\zeta^{(M} V^{N)} (u^R \xi_R)$. Note that even thought $\tilde{S}_{e,3}$ is zero on a torsionless background, its variation does lead to non-trivial constitutive relations.

Noting the identity

$$\left(\epsilon^{MNRST} V_R u_S \zeta_T\right)\left(\epsilon_{MKLOP} V^L u^O \zeta^P\right) = \zeta^2 P_\zeta{}^N{}_K, \tag{5.36}$$

we can define a new basis for one-derivative vectors

$$\begin{pmatrix} V_1'^M \\ V_2'^M \\ V_3'^M \end{pmatrix} = \begin{pmatrix} V_1^M \\ V_2^M \\ V_3^M \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{V}_1'^M \\ \tilde{V}_2'^M \\ \tilde{V}_3'^M \end{pmatrix}, \qquad \kappa'_{ij} = \kappa_{ij} + k_{ij}, \tag{5.37}$$

$\mathfrak{D}_0$	$T^{\mu\nu}$	J^μ	K
$-T\beta_{11}\begin{pmatrix} P_\zeta^{MN}P_\zeta^{RS} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{11}\,P_\zeta^{MN}S_1$		
$-2T\beta_{(12)}\begin{pmatrix} P_\zeta^{MN}\zeta^{(R}V^{S)}+P_\zeta^{RS}\zeta^{(M}V^{N)} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{(12)}P_\zeta^{MN}S_2$ $-2\beta_{(12)}\zeta^{(M}V^{N)}S_1$		
$-T\beta_{(13)}\begin{pmatrix} P_\zeta^{MN}\zeta^R\zeta^S+P_\zeta^{RS}\zeta^M\zeta^N & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{(13)}P_\zeta^{MN}S_3$ $-\beta_{(13)}\zeta^M\zeta^N S_1$		
$-T\beta_{(14)}\begin{pmatrix} \cdot & P_\zeta^{MN}\zeta^R & \cdot \\ P_\zeta^{RS}\zeta^M & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{(14)}\,P_\zeta^{MN}S_4$	$-\beta_{(14)}\,\zeta^M S_1$	
$-T\beta_{(15)}\begin{pmatrix} \cdot & \cdot & P_\zeta^{MN} \\ \cdot & \cdot & \cdot \\ P_\zeta^{RS} & \cdot & \cdot \end{pmatrix}$	$-\beta_{(15)}\,P_\zeta^{MN}S_6$		$-\beta_{(15)}S_1$
$-4T\beta_{22}\begin{pmatrix} V^{(M}\zeta^{N)}\zeta^{(R}V^{S)} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\beta_{22}\,V^{(M}\zeta^{N)}S_2$		
$-2T\beta_{(23)}\begin{pmatrix} V^{(M}\zeta^{N)}\zeta^R\zeta^S+V^{(R}\zeta^{S)}\zeta^M\zeta^N & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\beta_{(23)}V^{(M}\zeta^{N)}S_3$ $-\beta_{(23)}\zeta^M\zeta^N S_2$		
$-2T\beta_{(24)}\begin{pmatrix} \cdot & V^{(M}\zeta^{N)}\zeta^R & \cdot \\ V^{(R}\zeta^{S)}\zeta^M & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\beta_{(24)}\,V^{(M}\zeta^{N)}S_4$	$-\beta_{(24)}\,\zeta^M S_2$	
$-2T\beta_{(25)}\begin{pmatrix} \cdot & \cdot & V^{(M}\zeta^{N)} \\ \cdot & \cdot & \cdot \\ V^{(R}\zeta^{S)} & \cdot & \cdot \end{pmatrix}$	$-2\beta_{(25)}\,V^{(M}\zeta^{N)}S_6$		$-\beta_{(25)}S_2$
$-T\beta_{33}\begin{pmatrix} \zeta^M\zeta^N\zeta^R\zeta^S & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{33}\,\zeta^M\zeta^N S_3$		
$-T\beta_{(34)}\begin{pmatrix} \cdot & \zeta^M\zeta^N\zeta^R & \cdot \\ \zeta^R\zeta^S\zeta^M & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{(34)}\,\zeta^{(M}\zeta^{N)}S_4$	$-\beta_{(34)}\,\zeta^M S_3$	
$-T\beta_{(35)}\begin{pmatrix} \cdot & \cdot & \zeta^M\zeta^N \\ \cdot & \cdot & \cdot \\ \zeta^R\zeta^S & \cdot & \cdot \end{pmatrix}$	$-\beta_{(35)}\,\zeta^M\zeta^N S_6$		$-\beta_{(35)}S_3$
$-T\beta_{44}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \zeta^M\zeta^R & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$		$-\beta_{44}\,\zeta^M S_4$	
$-T\beta_{(45)}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \zeta^M \\ \cdot & \zeta^R & \cdot \end{pmatrix}$		$-\beta_{(45)}\zeta^M S_5$	$-\beta_{(45)}S_4$
$-T\beta_{55}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}$			$-\beta_{55}S_5$
$-4T\kappa_{11}\begin{pmatrix} V^{(M}P_\zeta^{N)(R}V^{S)} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$2\kappa_{11}\,V^{(M}V_1^{N)}$		
$-4T\kappa_{(12)}\begin{pmatrix} V^{(M}P_\zeta^{N)(R}\zeta^{S)}+\zeta^{(M}P_\zeta^{N)(R}V^{S)} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\kappa_{(12)}V^{(M}V_2^{N)}$ $-2\kappa_{(12)}\zeta^M V_1^N$		
$-2T\kappa_{(13)}\begin{pmatrix} \cdot & V^{(M}P_\zeta^{N)R} & \cdot \\ V^{(R}P_\zeta^{S)M} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\kappa_{(13)}V^{(M}V_3^{N)}$	$-\kappa_{(13)}V_1^M$	
$-4T\kappa_{22}\begin{pmatrix} V^{(M}P_\zeta^{N)(R}V^{S)} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$2\kappa_{22}\,\zeta^{(M}V_2^{N)}$		
$-2T\kappa_{(23)}\begin{pmatrix} \cdot & \zeta^{(M}P_\zeta^{N)R} & \cdot \\ \zeta^{(R}P_\zeta^{S)M} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\kappa_{(23)}\zeta^{(M}V_3^{N)}$	$-\kappa_{(23)}V_2^M$	
$-T\kappa_{33}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & P_\zeta^{MR} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$		$-\kappa_{33}V_3^M$	
$4T\tilde\kappa_{[12]}\begin{pmatrix} V^{(M}\tilde\epsilon^{N)(R}\zeta^S+\zeta^{(M}\tilde\epsilon^{N)(R}V^S & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\tilde\kappa_{[12]}V^{(M}\tilde V_2^{N)}$ $+2\tilde\kappa_{[12]}\zeta^{(M}\tilde V_1^{N)}$		
$2T\tilde\kappa_{[13]}\begin{pmatrix} \cdot & V^{(M}\tilde\epsilon^{N)R} & \cdot \\ -V^{(R}\tilde\epsilon^{S)M} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\tilde\kappa_{[13]}V^{(M}\tilde V_3^{N)}$	$2\tilde\kappa_{[13]}\tilde V_1^M$	
$2T\tilde\kappa_{[23]}\begin{pmatrix} \cdot & \zeta^{(M}\tilde\epsilon^{N)R} & \cdot \\ -\zeta^{(R}\tilde\epsilon^{S)M} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\tilde\kappa_{[23]}\zeta^{(M}\tilde V_3^{N)}$	$2\tilde\kappa_{[23]}\tilde V_2^M$	
$-T\eta\begin{pmatrix} P_\zeta^{M\langle R}P_\zeta^{S\rangle N} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\eta\,\sigma_\zeta^{MN}$		

Table 5.6: One-derivative Class D constitutive relations for a $(4+1)$-dimensional null superfluid. We have defined $\tilde\epsilon^{MN}=\epsilon^{MNRST}V_R u_S\zeta_T$. We have also included $\beta_{55}=\alpha/T$ in K for completeness.

$\overline{\mathfrak{D}}_0$	$T^{\mu\nu}$	J^μ	K
$-2T\beta_{[12]} \begin{pmatrix} P_\zeta^{MN}\zeta^{(R}V^{S)} - P_\zeta^{RS}\zeta^{(M}V^{N)} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{[12]}P_\zeta^{MN}S_2$ $+2\beta_{[12]}\zeta^{(M}V^{N)}S_1$		
$-T\beta_{[13]} \begin{pmatrix} P_\zeta^{MN}\zeta^R\zeta^S - P_\zeta^{RS}\zeta^M\zeta^N & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{[13]}P_\zeta^{MN}S_3$ $+\beta_{[13]}\zeta^M\zeta^N S_1$		
$-T\beta_{[14]} \begin{pmatrix} \cdot & P_\zeta^{MN}\zeta^R & \cdot \\ -P_\zeta^{RS}\zeta^M & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{(14)}\, P_\zeta^{MN}S_4$	$\beta_{[14]}\, \zeta^M S_1$	
$-T\beta_{[15]} \begin{pmatrix} \cdot & \cdot & P_\zeta^{MN} \\ \cdot & \cdot & \cdot \\ -P_\zeta^{RS} & \cdot & \cdot \end{pmatrix}$	$-\beta_{[15]}\, P_\zeta^{MN}S_6$		$\beta_{[15]}S_1$
$-2T\beta_{[23]} \begin{pmatrix} V^{(M}\zeta^{N)}\zeta^R\zeta^S - V^{(R}\zeta^{S)}\zeta^M\zeta^N & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\beta_{[23]}V^{(M}\zeta^{N)}S_3$ $+\beta_{[23]}\zeta^M\zeta^N S_2$		
$-2T\beta_{[24]} \begin{pmatrix} \cdot & V^{(M}\zeta^{N)}\zeta^R & \cdot \\ -V^{(R}\zeta^{S)}\zeta^M & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\beta_{[24]}\, V^{(M}\zeta^{N)}S_4$	$\beta_{[24]}\, \zeta^M S_2$	
$-2T\beta_{[25]} \begin{pmatrix} \cdot & \cdot & V^{(M}\zeta^{N)} \\ \cdot & \cdot & \cdot \\ -V^{(R}\zeta^{S)} & \cdot & \cdot \end{pmatrix}$	$-2\beta_{[25]}\, V^{(M}\zeta^{N)}S_6$		$\beta_{[25]}S_2$
$-T\beta_{[34]} \begin{pmatrix} \cdot & \zeta^M\zeta^N\zeta^R & \cdot \\ -\zeta^R\zeta^S\zeta^M & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\beta_{[34]}\, \zeta^{(M}\zeta^{N)}S_4$	$\beta_{[34]}\, \zeta^M S_3$	
$-T\beta_{[35]} \begin{pmatrix} \cdot & \cdot & \zeta^M\zeta^N \\ \cdot & \cdot & \cdot \\ -\zeta^R\zeta^S & \cdot & \cdot \end{pmatrix}$	$-\beta_{[35]}\, \zeta^M\zeta^N S_6$		$\beta_{[35]}S_3$
$-T\beta_{[45]} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \zeta^M \\ \cdot & -\zeta^R & \cdot \end{pmatrix}$		$-\beta_{[45]}\zeta^M S_5$	$\beta_{[45]}S_4$
$-4T\kappa_{[12]} \begin{pmatrix} V^{(M}P_\zeta^{N)(R}\zeta^{S)} - \zeta^{(M}P_\zeta^{N)(R}V^{S)} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\kappa_{[12]}V^{(M}V_2^{N)}$ $+2\kappa_{[12]}\zeta^M V_1^N$		
$-2T\kappa_{[13]} \begin{pmatrix} \cdot & V^{(M}P_\zeta^{N)R} & \cdot \\ -V^{(R}P_\zeta^{S)M} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\kappa_{[13]}V^{(M}V_3^{N)}$	$\kappa_{[13]}V_1^M$	
$-2T\kappa_{[23]} \begin{pmatrix} \cdot & \zeta^{(M}P_\zeta^{N)R} & \cdot \\ -\zeta^{(R}P_\zeta^{S)M} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\kappa_{(23)}\zeta^{(M}V_3^{N)}$	$\kappa_{[23]}V_2^M$	
$4T\tilde{\kappa}_{11} \begin{pmatrix} V^{(M}\tilde{\epsilon}^{N)(R}V^S & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\tilde{\kappa}_{11}V^{(M}\tilde{V}_1^{N)}$		
$4T\tilde{\kappa}_{(12)} \begin{pmatrix} V^{(M}\tilde{\epsilon}^{N)(R}\zeta^S - \zeta^{(M}\tilde{\epsilon}^{N)(R}V^S & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\tilde{\kappa}_{(12)}V^{(M}\tilde{V}_2^{N)}$ $-2\tilde{\kappa}_{(12)}\zeta^{(M}\tilde{V}_1^{N)}$		
$2T\tilde{\kappa}_{(13)} \begin{pmatrix} \cdot & V^{(M}\tilde{\epsilon}^{N)R} & \cdot \\ V^{(R}\tilde{\epsilon}^{S)M} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\tilde{\kappa}_{(13)}V^{(M}\tilde{V}_3^{N)}$	$-2\tilde{\kappa}_{(13)}\tilde{V}_1^M$	
$4T\tilde{\kappa}_{22} \begin{pmatrix} \zeta^{(M}\tilde{\epsilon}^{N)(R}\zeta^S & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\tilde{\kappa}_{22}\zeta^{(M}\tilde{V}_2^{N)}$		
$2T\tilde{\kappa}_{(23)} \begin{pmatrix} \cdot & \zeta^{(M}\tilde{\epsilon}^{N)R} & \cdot \\ \zeta^{(R}\tilde{\epsilon}^{S)M} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-2\tilde{\kappa}_{(23)}\zeta^{(M}\tilde{V}_3^{N)}$	$-2\tilde{\kappa}_{(23)}\tilde{V}_2^M$	
$T\tilde{\kappa}_{33} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \tilde{\epsilon}^{MR} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$		$-2\tilde{\kappa}_{33}\tilde{V}_3^M$	
$T\tilde{\eta} \begin{pmatrix} P_\zeta^{T(M}\tilde{\epsilon}^{N)(R}P_\zeta^{S)}{}_T & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$-\tilde{\eta}\,\tilde{\sigma}_\zeta^{MN}$		

Table 5.7: One-derivative Class $\overline{\mathrm{D}}$ constitutive relations for a $(4+1)$-dimensional null superfluid. We have defined $\tilde{\epsilon}^{MN} = \epsilon^{MNRST}V_R u_S \zeta_T$.

where

$$[a_{ij}] = \begin{pmatrix} 0 & \frac{\tilde{\kappa}_{[12]}}{\kappa_{11}} & \frac{\kappa_{11}\left(\kappa_{22}\tilde{\kappa}_{[13]}-\kappa_{(12)}\tilde{\kappa}_{[23]}\right)-\tilde{\kappa}_{[12]}\left(\kappa_{(12)}\kappa_{(13)}+\zeta^M\zeta_M\tilde{\kappa}_{[12]}\tilde{\kappa}_{[13]}\right)}{\kappa_{11}\left(\kappa_{11}\kappa_{22}-\kappa_{(12)}^2-\zeta^M\zeta_M\tilde{\kappa}_{[12]}^2\right)} \\ 0 & 0 & \frac{\kappa_{11}\tilde{\kappa}_{[23]}-\kappa_{(12)}\tilde{\kappa}_{[13]}+\kappa_{(13)}\tilde{\kappa}_{[12]}}{\kappa_{11}\kappa_{22}-\kappa_{(12)}^2-\zeta^M\zeta_M\tilde{\kappa}_{[12]}^2} \\ 0 & 0 & 0 \end{pmatrix},$$

$$[k_{ij}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\zeta^M\zeta_M\frac{(\tilde{\kappa}_{[12]})^2}{\kappa_{11}} & -\zeta^M\zeta_M\frac{\tilde{\kappa}_{[12]}\tilde{\kappa}_{[13]}}{\kappa_{11}} \\ 0 & -\zeta^M\zeta_M\frac{\tilde{\kappa}_{[12]}\tilde{\kappa}_{[13]}}{\kappa_{11}} & -\zeta^M\zeta_M\left(\frac{(\tilde{\kappa}_{[13]})^2}{\kappa_{11}} + \frac{\left(\kappa_{11}\tilde{\kappa}_{[23]}-\kappa_{(12)}\tilde{\kappa}_{[13]}+\kappa_{(13)}\tilde{\kappa}_{[12]}\right)^2}{\kappa_{11}\left(\kappa_{11}\kappa_{22}-\kappa_{(12)}^2-\zeta^M\zeta_M\tilde{\kappa}_{[12]}^2\right)}\right) \end{pmatrix}. \tag{5.38}$$

The rationale is that in this basis, the quadratic form Δ can be arranged into a form which can be made explicitly positive semi-definite

$$T\Delta = \sum_{i,j=1}^{5} S_i\beta_{(ij)}S_j + \sum_{i,j=1}^{3} V_i'^M\kappa'_{(ij)}V'_{j,M} + \eta\sigma^{MN}\sigma_{MN}. \tag{5.39}$$

Given $T \geq 0$, the condition $\Delta \geq 0$ implies that $\eta \geq 0$ and the matrices $[\beta_{(ij)}]_{5\times 5}$ and $[\kappa'_{(ij)}]_{3\times 3}$ have all non-negative eigenvalues. This gives us 9 inequalities among the 25 dissipative transport coefficients, while 16 remain completely arbitrary.

This finishes our discussion of the first order Galilean superfluid dynamics. Due to their technical form, we have left the explicit constitutive relations in their null fluid form. We can always perform the null reduction prescribed in section 4.2 and express them in the Newton-Cartan language. For reference, we have null reduced all the one-derivative superfluid data in table 5.4. The explicit null reduction of the constitutive relations can be found in our paper [4].

5.3 Galilean fluid surfaces

For the final example of this thesis, we consider the breaking of the spacetime translational symmetry to form surfaces in a Galilean fluid. The discussion presented here is taken from [6], and is a straight forward generalisation of our relativistic discussion in section 3.3. We introduce a Goldstone mode $f(x)$ corresponding to a broken momentum generator, with the symmetry transformation $\delta_\mathscr{X} f = \chi^M\partial_M f$. We require that $V^M\partial_M f = 0$. We also introduce a distribution functional $\theta(f)$, with properties similar to section 3.3: its derivative $\theta'(f)$ is strictly positive and is only supported in a thin band around $f = 0$; all the f-dependence in the constitutive relations comes via $\theta(f)$; the function $f \sim \mathcal{O}(\partial^{-1})$, while the distribution $\theta(f) \sim \mathcal{O}(\partial^0)$. Derivatives of $\theta(f)$ can be characterised in terms of the derivatives of the unit normal vector

$$z_M = -\frac{\partial_M f}{\sqrt{g^{MN}\partial_M f\partial_N f}}, \tag{5.40}$$

along with the distributions

$$\tilde{\delta}^{(n)}(f) = (-)^{n+1} z^{M_1} \ldots z^{M_{n+1}} \mathrm{D}_{M_1} \ldots \mathrm{D}_{M_{n+1}} \theta(f). \tag{5.41}$$

Denoting the equation of motion for f by $Y/\sqrt{g^{MN}\partial_M f \partial_N f} \approx 0$, we can write the Galilean adiabaticity equation as

$$\mathrm{D}_M N^M - \mathrm{N}^{\perp}_{\mathrm{H}} = \frac{1}{2} T^{MN} \delta_{\mathscr{B}} g_{MN} + J^M \delta_{\mathscr{B}} A_M + \frac{Y \delta_{\mathscr{B}} f}{\sqrt{g^{MN}\partial_M f \partial_N f}} + \Delta, \qquad \Delta \geq 0. \tag{5.42}$$

In an exact correspondence with the relativistic results in section 3.3, up to one-derivative order, this equation allows a set of Class $\mathrm{H_S}$ constitutive relations parametrised by

$$\mathcal{N} = \theta(f) P_{\mathrm{in}}(T, \mu, \mu_{\mathrm{m}}) + \bar{\theta}(f) P_{\mathrm{out}}(T, \mu, \mu_{\mathrm{m}}) - \tilde{\delta}(f) \gamma(T, \mu, \mu_{\mathrm{m}}). \tag{5.43}$$

Here $\bar{\theta}(f) = 1 - \theta(f)$. In Class D we have a term like $Y \sim -\alpha \theta'(f) \delta_{\mathscr{B}} f$ for some non-negative coefficient α. Plus, following section 5.1.2, we have two copies of Class D and $\overline{\mathrm{D}}$ constitutive relations η, ζ, κ, σ, κ_Q, and $\overline{\kappa}_Q$, one on either side of the surface, and one copy of Class A and Class $\mathrm{H_V}$ constants. Together, they imply a set of constitutive relations

$$\begin{aligned}
T^{MN} &= \theta(f) T^{MN}_{\mathrm{in}} + \bar{\theta}(f) T^{MN}_{\mathrm{out}} \\
&\qquad + \tilde{\delta}(f) \Big(R_{\mathrm{sur}} u^M u^N + 2 \left(E_{\mathrm{sur}} - \gamma\right) u^{(M} V^{N)} - \gamma \left(g^{MN} - z^M z^N\right) \Big) + \mathcal{O}(\partial^2), \\
J^M &= \theta(f) J^M_{\mathrm{in}} + \bar{\theta}(f) J^M_{\mathrm{out}} + \tilde{\delta}(f) Q_{\mathrm{sur}} u^M + \mathcal{O}(\partial^2), \\
Y &= \tilde{\delta}(f) \left(\frac{\alpha}{T} u^M z_M + (P_{\mathrm{in}} - P_{\mathrm{out}}) - \mathrm{D}_M \left(\gamma z^M\right) \right) + \mathcal{O}(\partial^2).
\end{aligned} \tag{5.44}$$

The bulk contributions $T^{MN}_{\mathrm{in/out}}$ and $J^M_{\mathrm{in/out}}$ to the energy-momentum tensor and charge current are taken directly from section 5.1.2. In addition, there are surface contributions to the constitutive relations, which are basically ideal Galilean fluids living at the boundary, with the negative of surface tension acting as a pressure.

Upon null reduction they imply the Galilean fluid constitutive relations in the Newton-Cartan language

$$\begin{aligned}
\rho^\mu &= \theta(f) \rho^\mu_{\mathrm{in}} + \bar{\theta}(f) \rho^\mu_{\mathrm{out}} + \tilde{\delta}(f) R_{\mathrm{sur}} u^\mu + \mathcal{O}(\partial^2), \\
\epsilon^\mu &= \theta(f) \epsilon^\mu_{\mathrm{in}} + \bar{\theta}(f) \epsilon^\mu_{\mathrm{out}} + \tilde{\delta}(f) \left[\left(E_{\mathrm{sur}} + \frac{1}{2} R_{\mathrm{sur}} u^a u_a \right) u^\mu - \gamma \left(\bar{u}^\mu + z^\mu v^\nu z_\nu\right) \right] + \mathcal{O}(\partial^2), \\
p^\mu{}_a &= \theta(f) p_{\mathrm{in}}{}^\mu{}_a + \bar{\theta}(f) p_{\mathrm{out}}{}^\mu{}_a + R_{\mathrm{sur}} u^\mu - \gamma \left(e_a{}^\mu - z^\mu z_a\right) + \mathcal{O}(\partial^2) \\
j^\mu &= \theta(f) j^\mu_{\mathrm{in}} + \bar{\theta}(f) j^\mu_{\mathrm{out}} + \tilde{\delta}(f) Q_{\mathrm{sur}} u^\mu + \mathcal{O}(\partial^2).
\end{aligned} \tag{5.45}$$

Note that $z^\mu = h^{\mu\nu} z_\nu$ along with the normalisation condition $z_\mu z_\nu h^{\mu\nu} = 1$. On the other hand, we get the Young-Laplace equation

$$u^\mu z_\mu \approx -\frac{T}{\alpha} \Big(\Delta P - h^{\mu\nu} \tilde{\mathrm{D}}_\mu \left(\gamma z_\nu\right) \Big) + \mathcal{O}(\partial), \qquad \text{where} \quad \Delta P = P_{\mathrm{in}} - P_{\mathrm{out}}, \tag{5.46}$$

which is reminiscent of its relativistic incarnation. We see that the motion of the surface is governed by the balance of pressures inside and outside, and the surface tension term.

This finishes our discussion of Galilean fluids with surfaces. The generalisation of these results to Galilean superfluids is quite straight forward and has been discussed in [6].

6 | Outlook

In this thesis, we discussed the principles of relativistic and Galilean hydrodynamics. We built upon the fundamental considerations of symmetries, thermal field theories, and thermodynamics, and presented a universal framework for hydrodynamics that allows for arbitrary gapless modes in its spectrum. This brings together a variety of otherwise distinct hydrodynamic treatments like superfluid dynamics, surface fluid dynamics, magnetohydrodynamics, and obviously ordinary fluid dynamics. We argued that the time-evolution of these gapless modes can also be derived from within the hydrodynamic framework by a suitable generalisation of the second law of thermodynamics to off-shell configurations. This provides a neat hydrodynamic argument to derive the Josephson equation in superfluid dynamics and Young-Laplace equation in surface fluid dynamics. Our treatment of hydrodynamics also includes an arbitrary spin current and background torsion, which has not been discussed in full generality in the literature before.

Using this framework, we presented an all-order analysis of the second law of thermodynamics and classified the most generic hydrodynamic transport compatible with it. We found that the second law imposes some strict constraints in the hydrostatic sector, by forcing the constitutive relations to be characterised by a free energy density (Class $\mathrm{H_S}$) and a finite set of constants (Class A and $\mathrm{H_V}$). In the non-hydrostatic sector, however, we found the second law to be much more lenient; it classifies the hydrodynamic transport into Class D and Class $\overline{\mathrm{D}}$, but only imposes some inequalities on the leading order Class D transport coefficients and none thereafter. Prior to this work, a similar classification analysis had been done in the literature for the restricted case of ordinary fluid dynamics [52]. Although our classification scheme draws heavily from that of [52] in the hydrostatic sector, the two are quite distinct in the non-hydrostatic sector. Most notably, the classification presented in this thesis is non-redundant, i.e. there is no overlap between various classes, and it purifies the true dissipative transport from the mere redundancies (Class S) in the choice of an entropy current that satisfies the second law for a given set of constitutive relations.

Apart from reproducing some known results in hydrodynamics, we applied the ideas discussed in this thesis to study some novel hydrodynamic systems. We presented a theory of non-Abelian superfluid dynamics with a partially broken (semisimple) Lie group of internal symmetries. We introduced a pair of projection operators that allowed us to define a superfluid velocity corresponding to the Goldstone modes valued in the Lie-algebra quotient, irrespective of the details of the explicit symmetry breaking. Expressed in terms of these projectors, we illustrated that the non-Abelian superfluid dynamics can be made structurally similar to its Abelian counterpart. We derived the non-Abelian generalisation of the Josephson equation, which, similar to the Abelian case, determines the time derivatives of the Goldstone modes in terms of the corresponding Lie-algebra components of the chemical potential. Recently, non-Abelian superfluids have started to gain some attention in the literature regarding the modelling of p-wave superfluidity observed in liquid ^{3}He [97]. It will be interesting to see if the results found in this thesis have an analogue in these physical systems.

We also studied how the hydrodynamic constitutive relations are modified near a surface or an interface in a fluid. We introduced a new gapless hydrodynamic mode, called a shape-field, to characterise such surfaces and used its dynamics to reproduce the Young-Laplace equation. We worked out the hydrodynamic constitutive relations up to one-derivative order, which amounts to zero-derivative order on the surface, and identified the well-known transport coefficient, surface tension, in the surface thermodynamics. Hydrodynamic configurations with surfaces are quite interesting as they can be used to model fluids undergoing a phase transition, like the fluid-superfluid phase transition in liquid helium or the confinement-deconfinement phase transition in the quark-gluon plasma. Further exploration in this direction, perhaps holographic, will be quite intriguing.

In the introduction of this thesis, we briefly outlined how the theory of magnetohydrodynamics (MHD) can also be incorporated into the same framework by allowing for a dynamical U(1) gauge field as an additional gapless mode in the hydrodynamic description. Although this preliminary analysis checks out quite naturally, we have not considered MHD in rigorous detail in this thesis. As a natural next step, we should verify that the calculation does indeed work out at arbitrarily high orders in the derivative expansion without any surprises. As an interesting aside, note that the dynamical equation of MHD (1.51) forces us to set the charge density to be an order one quantity in the derivative expansion, which algebraically determines the associated chemical potential in terms of the other fields in the theory. The same equation is also present in the conventional formulation of MHD as well; see e.g. [54]. However, the chemical potential is still typically taken to be an independent hydrodynamic variable in MHD, which is puzzling. It will be interesting to see if the off-shell formalism of magnetohydrodynamics can shed some light on this issue.

On another front, in a recent paper [60], an alternative interpretation of MHD has been proposed. Instead of introducing a dynamical gauge field, the authors considered hydrodynamics with a global 1-form symmetry corresponding to the conservation of magnetic field lines and reproduced the spectrum of MHD. It was realised in [8], however, that there are some issues with defining hydrostatic configurations in these theories. In particular, unlike regular hydrodynamics, in the presence of a higher-form symmetry, the second law of thermodynamics does not seem to be sufficient to guarantee the hydrostatic principle. Since the off-shell formalism of hydrodynamics detailed in this thesis renders the connection between the second law and the hydrostatic principle quite transparent, we would like to see if it can be utilised to resolve this discrepancy.

In the second part of this thesis, we offered a new viewpoint of Galilean hydrodynamics. Realising that a generic Galilean theory can be covariantly coupled to a one-higher dimensional null background, we formalised hydrodynamics on such null backgrounds from an axiomatic standpoint. The fluids thus obtained, which we call null fluids, are a one-higher dimensional relativistic embedding of Galilean fluids, which manifest all the Galilean symmetries, especially boosts, in terms of a Poincaré invariant structure. They are essentially relativistic fluids coupled to a spacetime background carrying a null isometry. Due to the anisotropy introduced by this additional background vector field, null/Galilean fluids are characterised by many more transport coefficients compared to their relativistic cousins. One way to understand this distinction is to note that the Galilean fluids carry an additional conserved current, i.e. mass current, in their spectrum, with its own dynamical chemical potential and background gauge field. Correspondingly, we have additional constitutive relations to write down and additional fields to generate the associated tensor structures, which eventually leads to the increase in the number of allowed transport coefficients.

Owing to our handle on the Poincaré symmetry, relativistic fluids are typically much better understood in the literature compared to their Galilean counterparts. Null fluids can potentially bridge this gap in our understanding by providing a mechanism to import all the exotic relativistic machinery at our disposal directly into Galilean hydrodynamics. We witnessed two non-trivial applications of this idea in this thesis. Firstly, we imported the classification scheme of relativistic hydrodynamics into Galilean hydrodynamics and used it to perform an all-order analysis of the second law of thermodynamics. Similar to the relativistic case, we found that the second law imposes some strict constraints in the hydrostatic sector, while only requires some leading-order transport coefficients to be non-negative in the non-hydrostatic sector. We studied the explicit application of this to ordinary Galilean fluids, Galilean superfluids, and surfaces in Galilean fluids. Secondly, we used the preexisting transgression machinery of relativistic hydrodynamics [94] to work out the effects of anomalies on the Galilean hydrodynamic transport. We also used this language to classify the set of constants, called transcendental anomalies, which are left undetermined by the second law of thermodynamics. We found that Galilean hydrodynamics admits many more of these constants compared to the relativistic case. There has been some work in the literature which associates these undetermined constants in relativistic hydrodynamics to anomaly coefficients by demanding the consistency of Euclidean vacuum [87]. We are not aware if a similar argument exists for Galilean hydrodynamics as well.

During this construction, we devised an anomaly inflow mechanism for Galilean field theories using null backgrounds, classifying the possible 't Hooft anomalies. The anomalies we found only affect the rotations and internal symmetries, while leaving the space-time translations, mass conservation, and Galilean boost symmetries non-anomalous. It is interesting to note that the Galilean anomaly polynomial is structurally equivalent to the relativistic anomaly polynomial, and hence the number of anomaly coefficients on both sides match. Owing to this, the structure of Hall currents that enter the conservation laws is also quite similar in both the cases. Hence the results we have obtained promise to be the genuine non-relativistic anomalies and not just a mathematical manifestation of the Galilean invariance. That being said, we still need to explicitly construct a Galilean field theory that exhibits these anomalies, which we leave for the future explorations.

As we discussed in the introduction, a fundamental motivation for the continued research interest in hydrodynamics is to get insights into the physics of out-of-equilibrium thermal field theories, especially the irreversible dissipative phenomena. A part of this program is to write down a Wilsonian effective action for hydrodynamics, which should provide a first principle understanding of the dissipative hydrodynamic transport. The off-shell formalism of hydrodynamics discussed here provides a natural starting point to frame these questions. Much progress has already been made towards writing down a Schwinger-Keldysh effective action for ordinary fluid dynamics [68, 71]. As a prospective direction, it will be interesting to see if these constructs can be extended to include the arbitrary gapless modes discussed in this thesis, thereby providing an extension to more generic hydrodynamic treatments like superfluid dynamics and magnetohydrodynamics. For now, we leave these ambitious ventures open for speculation.

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